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# HETEROGENEOUS TREATMENT EFFECTS ESTIMATION: WHEN MACHINE LEARNING MEETS MULTIPLE TREATMENT REGIME

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## ABSTRACT

In many scientific and engineering domains, inferring the effect of treatment and exploring its heterogeneity is crucial for optimization and decision making. In addition to Machine Learning based models (e.g. Random Forests or Neural Networks), many meta-algorithms have been developed to estimate the Conditional Average Treatment Effect (CATE) function in the binary setting, with the main advantage of not restraining the estimation to a specific supervised learning method. However, this task becomes more challenging when the treatment is not binary. In this paper, we investigate the Rubin Causal Model under the multi-treatment regime and we focus on estimating heterogeneous treatment effects. We generalize *Meta-learning* algorithms to estimate the CATE for each possible treatment value. Using synthetic and semi-synthetic simulation datasets, we assess the quality of each meta-learner in observational data, and we highlight in particular the performances of the X-learner.

**Keywords** Machine Learning · Causal Inference · Multiple Treatments · Heterogeneous Effects.

## 1 Introduction

With the rapid development of Machine Learning (ML) and its efficiency in predicting outcomes, the question of counterfactual prediction "*what would happen if ?*" arises. Engineers may want to know how the outcome (e.g. production) would be affected when a parameter is changed to a specific value. It will help them personalize the parameter at efficient levels and optimize the outcome, either on average or on a smaller scale (sub-groups of units). Recently, many companies have relied on ML models to find the optimal intervention strategy. Yet, the results are not satisfactory. Indeed, these models do not account for other impacting effects (One-At-a-Time approach) and cannot distinguish between correlations and causal relationships in the data. For instance, Li et al. (2016) have quantified the impact of increasing number of stages on Enhanced Geothermal System (EGS) performance using sensitivity analysis. The results are, however, completely different from what ML models predict.

Based on the Potential Outcomes theory (Neyman, 1923; Rubin, 1974), epidemiologists and statisticians developed a set of statistical tools to make causal inference and estimate the effects of a treatment on the outcome whether on average among the whole population or inside different sub-groups. They have been successfully applied in many fields such as medicine (Foster et al., 2011; Shalit et al., 2017; Alaa and van der Schaar, 2017), economics (Knaus et al., 2020;

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Jacob, 2021), public policy (Imai and Strauss, 2011; Turney and Wildeman, 2015), and advertising/marketing (Bottou et al., 2013; Li et al., 2015; Diemert et al., 2018). Nevertheless, they are still unfamiliar and seldom used in industrial applications.

Further, most existing methods and studies are limited to the setting of a binary treatment, whereas in many real-world applications, the treatment variable can take multiple values. In some cases, it would be helpful to give an in-depth analysis of the impact of the treatment across its possible levels (doses) instead of just considering a binary scenario where the treatment is either assigned or not. In addition, the heterogeneity of effects may provide valuable information regarding the effectiveness of this treatment and help individuals, companies or governments to personalize their policies and strategies.

Finally, randomized controlled trials (RCT) are not always conducted, and the ground truth of treatment effects cannot be observed and is rarely available. This fact makes Heterogeneous Treatment Effects estimation different from a standard supervised learning problem (Alaa and van der Schaar, 2018). Therefore, it is challenging to assess treatment effect estimators performance and select the best model with standard point-wise error metrics such as Mean Squared Errors.

In this work, we study the problem of estimating Heterogeneous Treatment Effects, also known as Conditional Average Treatment Effects (CATEs), when the treatment is multi-valued. We put our focus on establishing *plug-in estimators*, also referred as *meta-learners* (Künzel et al., 2019) or *model-agnostic algorithms* (Curth and van der Schaar, 2021a), that estimate the CATE with any supervised ML model. This task can be achieved either by modelling the CATE directly in one step or two steps: by decomposing it into regularized regression problems or by addressing a minimization problem with respect to an appropriate loss function.

**Contributions.** The main contributions of this paper can be summed up as follows. First, we generalize the notion of *meta-learners* in the multi-treatment setting to estimate CATEs. We propose adapted representations of the outcome (i.e. pseudo-outcome) to learn the CATE efficiently in observational studies. Second, using a generalization of Robinson (1988) decomposition, we propose an approach of R-learning, and we form an objective function that captures causal effect. Finally, we evaluate our meta-learners on synthetic datasets and a semi-synthetic dataset simulating a geothermal system. This dataset covers all possible scenarios and serves as a baseline to compare the obtained counterfactual predictions with the ground truth response.

The remainder of the paper is structured as follows. Section 2 introduces related work. Section 3 presents problem setup about the CATE estimation. In Section 4 we provide a theoretical setup of CATEs estimators (Meta-learners) when the treatment is multi-valued. We present empirical evaluation results in Section 5. Finally, we conclude in Section 6 with a discussion.

## 2 Related work

**Meta-learners for CATEs estimation.** The recent interest and advances in CATEs estimation have led to the development of numerous algorithms (see Caron et al. (2021) for a review). Some of them incorporate ML through modified models (Hill, 2011; Hahn et al., 2020; Alaa and van der Schaar, 2017; Athey et al., 2019; Yoon et al., 2018; Fan et al., 2020) while others, known as *meta-learners*, do not require a specific ML method. The theory of *meta-learners* was initially introduced and discussed by Künzel et al. (2019) for the CATE estimation in the binary setting. Three learners have been considered in their work: the S-learner, the T-learner and the X-learner. Kennedy (2020) proposes the DR-learner (doubly robust) using the augmented inverse propensity weighted (AIPW) outcome to overcome the problem of model misspecifications when estimating nuisance functions (e.g. the propensity score and outcome models). Nie and Wager (2020) presents R-learner that estimates the CATE by minimizing the squared error loss based on orthogonalization with respect to both observed outcome and propensity score estimate. Curth and van der Schaar (2021a) consider PW-learner (propensity weighting learner), borrowed from the inverse probability weighting transformation as considered by Athey and Imbens (2016), and RA-learner (regression adjustment learner), which is an improved version of the X-learner. They also suggest categorization of meta-learners by dividing them, excluding the R-learner, into two categories: one-step plug-in estimators, which are learners that estimate two regression functions then estimate the CATE as a difference of these functions (T- and S-learners), and two-step plug-in learners (RA-, PW- and DR-learners), that estimate the previous nuisance functions then target the CATE directly by regressing a pseudo-outcome. They show that, under some conditions, two-step learners can attain the oracle rate.

**Multiples and continuous treatments.** Recent years have seen a growing interest in developing causal inference methods with multi-valued treatments using observational data. The theoretical work of Imbens (2000); Lechner (2001); Frölich (2002); Imai and Dyk (2004) extended the potential outcome framework and the propensity score in the general

treatment setting, including also continuous treatments. Nonetheless, most studies focus on the average treatment effect (Flores, 2007; Zhu et al., 2015; Saini et al., 2019; Colangelo and Lee, 2021; Nie et al., 2021) only a few of them handle heterogeneous treatment effects. Heiler and Knaus (2021) show that binarizing multi-treatments can lead to a spurious estimation of heterogeneous effects across different levels. Schwab et al. (2020), followed by Harada and Kashima (2021), considered graph-structured treatments and representation learning approaches to present a ML approach using neural networks to estimate individual dose-response curves for multi-treatments. Freshly, Kaddour et al. (2021) proposed Structured Intervention Networks (SIN), a representation learning that uses the generalized Robinson decomposition for estimating CATEs of structured treatments. This work is close to the R-learning approach in subsection 4.4.

### 3 Problem formulation

To address the problem of inferring causal effects when the treatment is not binary but has at least three possible values, we follow the potential outcome theory (Rubin-Neyman model) as extended to the multi-treatment regime by Imbens (2000); Lechner (2001); Frölich (2002); Imai and Dyk (2004) and we consider the following statistical problem.

Following the potential outcomes framework, we suppose the existence of  $Y(t)$ , the real-valued counterfactual outcome that would have been observed under treatment level  $t \in \mathcal{T} = \{t_1, \dots, t_K\}$ . We consider  $(\mathbf{X}, T, Y(t)_{t \in \mathcal{T}}) \sim \mathbb{P}$  where  $\mathbf{X} = (X^{(1)}, \dots, X^{(d)}) \in \mathbb{R}^d$  denotes a random vector of covariates and  $T$  denotes the treatment assignment random variable. We suppose finally that we observe data drawn from independent and identically distributed sample of  $n$  units  $D_{\text{obs},i} = (\mathbf{X}_i, T_i, Y_{\text{obs},i})$  distributed as  $(\mathbf{X}, T, Y_{\text{obs}})$  with  $Y_{\text{obs}} = Y(T)$  (consistency assumption).

Note that the distribution of the observed sample  $\mathcal{D}_{\text{obs}} = (D_{\text{obs},i})_{i=1}^n$  is not representative of the whole population due to *selection bias*, standing for the fact that covariates variables affect units' treatment assignment. This dependence is usually quantified by the generalized propensity score  $r(t, \mathbf{x}) := \mathbb{P}(T = t | \mathbf{X} = \mathbf{x})$  (Imbens, 2000). This parameter has the same balancing propriety as the classical Propensity Score (Rosenbaum and Rubin, 1983). It is widely used to remove confounding bias in observational studies to get the RCT setting where  $\mathbf{X}$  and  $T$  are independent.

We aim to infer the effect of the treatment  $T$  on the outcome  $Y$ . More precisely, we want to estimate the CATE between two levels of treatment  $T = t_1$  and  $T = t$ , defined as

$$\tau_t(\mathbf{x}) = \mathbb{E}(Y(t) - Y(t_1) | \mathbf{X} = \mathbf{x}), \quad (1)$$

which can be interpreted as is the expected treatment effect between levels  $T = t_1$  (defined as baseline treatment value) and  $t$  for an individual with covariates  $\mathbf{X} = \mathbf{x}$ .

Unfortunately, it is impossible to infer this effect directly. We observe only one potential outcome corresponding to the potential outcome receiving the treatment  $T$  (i.e. the real outcome) for every unit. All other potential outcomes are missing (inherently unobservable). This is known as the fundamental problem of causal inference (Holland, 1986). Consequently, to identify the causal effects from the observed sample data, we shall consider the following assumptions (Assumption 3.1 is unfortunately *untestable*)

**Assumption 3.1.** Unconfoundedness: The treatment mechanism is unconfounded given the observed covariates  $Y(t) \perp\!\!\!\perp \mathbf{1}\{T = t\} \mid \mathbf{X}$  for all  $t \in \mathcal{T}$ .

**Assumption 3.2.** Overlap: the probability of receiving the treatment given observed covariates is positive i.e. there exists  $r_{\min} > 0$  such that  $r_{\min} \leq r(t, \mathbf{x}) = \mathbb{P}(T = t | \mathbf{X} = \mathbf{x})$ .

With the previous assumptions, the expected potential outcome satisfies  $\mathbb{E}(Y(t) | \mathbf{X} = \mathbf{x}) = \mathbb{E}(Y_{\text{obs}} | T = t, \mathbf{X} = \mathbf{x})$  and the CATE can be identified as non-parametric estimation problem.

The problem of the CATE estimation can be seen as a non-parametric estimation problem. However, this paper does not consider the discussion about model-fitting and the best strategy for learning.

### 4 Meta-learners in the multi-treatment regime

To tackle the problem of estimating CATEs under multi-valued treatment, we generalize the notion of meta-learners to derive consistent estimators of the CATE. Moreover, all considered meta-learners below, except the R-learner, can support any supervised regression ML method (e.g. random forest, gradient boosting methods). These ML methods are called *base-learners* when applied to a meta-learner.

In the following, we follow a similar taxonomy of CATEs estimators as Curth and van der Schaar (2021a); Knaus et al. (2020). Namely, direct plug-in (one step) meta-learners, pseudo-outcome (two-step) meta-learners and Neyman-Orthogonality based learners (R-learner).

#### 4.1 Direct plug-in meta-learners

In this subsection, we present direct plug-in meta-learners, also known as *one-step* learners that estimate the CATE in (1) by targeting directly the observed data  $\mathcal{D}_{\text{obs}}$ .

**T-learner with multiple treatments.** The T-learning is naive approach of learning the CATE. It consists on estimating the *two* conditional response surfaces  $\mu_w(\mathbf{x}) = \mathbb{E}(Y(w) \mid \mathbf{X} = \mathbf{x})$  using  $\mathbf{S}_w = \{i, T_i = w\}$  for  $w \in \{t, t_1\}$  as in the binary case. Basically, the T-learning approach does not account for the interaction between treatment  $T$  and the outcome  $Y$  and create different models for different treatments.

Despite its naivety, the T-learning approach may suffer from selection bias (Curth and van der Schaar, 2021b), that is, when the outcome models  $\mu_w$  are estimated with respect to the wrong distribution of the training sample. To overcome this issue in the estimation of  $\mu_w$  while sampling  $(D_{\text{obs},i})_{i \in \mathbf{S}_w}$ , we use Importance Sampling (Hassanpour and Greiner, 2019), and we show the following proposition.

**Proposition 4.1.** *For a treatment level  $w \in \mathcal{T}$ , the expected squared error of the estimator  $\hat{\mu}_w$  on the outcome surface  $\mu_w$  satisfies :*

$$\begin{aligned} \mathbb{E}_{\mathbf{X} \sim \mathbb{P}(\cdot)} \left[ (\hat{\mu}_w(\mathbf{X}) - \mu_w(\mathbf{x}))^2 \right] = \\ \mathbb{E}_{\mathbf{X} \sim \mathbb{P}(\cdot \mid T=w)} \left[ \frac{\mathbb{P}(T=w)}{r(w, \mathbf{X})} (\hat{\mu}_w(\mathbf{X}) - \mu_w(\mathbf{X}))^2 \right]. \end{aligned}$$

The proposition 4.1 highlights the fact that  $\mu_w$  should be estimated by minimizing the expected squared error on the nominal *weighted* distribution.

Therefore, the T-learner the in multi-treatment setting can be built as follows

- For  $w \in \{t, t_1\}$ , sample  $(D_{\text{obs},i})_{i \in \mathbf{S}_w}$  and estimate the conditional response  $\hat{\mu}_w$  by minimizing the expected squared error of the estimator  $\hat{\mu}_w$ .
- Compute the CATE between two treatment levels  $t$  and  $t_1$  by  $\hat{\tau}_t^{(T)}(\mathbf{x}) = \hat{\mu}_t(\mathbf{x}) - \hat{\mu}_{t_1}(\mathbf{x})$ .

**S-learner with multiple treatments.** Using the identification of the CATE by assumptions (3.1)-(3.2) :  $\tau_t(\mathbf{x}) = \mathbb{E}(Y_{\text{obs}} \mid T = t, \mathbf{X} = \mathbf{x}) - \mathbb{E}(Y_{\text{obs}} \mid T = t_1, \mathbf{X} = \mathbf{x})$ . Therefore, instead of splitting the dataset and building separate models as in T-learning, one can consider a *single* model built from the whole dataset and define naturally the S-learner in case of the multi-treatment setting as

- Regress  $Y_{\text{obs}}$  on the treatment  $T$  and the covariates  $\mathbf{X}$  by a single model  $\hat{\mu}$  using  $\mathcal{D}_{\text{obs}}$ .
- Estimate the CATE between two treatment levels  $t$  and  $t_1$  by  $\hat{\tau}_t^{(S)}(\mathbf{x}) = \hat{\mu}(\mathbf{x}, t) - \hat{\mu}(\mathbf{x}, t_1)$ .

Obviously, including the treatment  $T$  as an input feature and sharing some information between covariates  $\mathbf{X}$  and  $T$  may provide better predictions. However, this result is conditioned by the ability of the regression model to capture and distinguish contributions of both  $\mathbf{X}$  and  $T$  on  $Y_{\text{obs}}$ .

*Remark.* Even though  $\mathbb{E}(Y(t) \mid \mathbf{X} = \mathbf{x}) = \mathbb{E}(Y_{\text{obs}} \mid T = t, \mathbf{X} = \mathbf{x})$ , it is very important to keep in mind the distinction between the S- and T-learning. The T-learner regresses on  $\mathbf{X}$  for  $t$  fixed whereas the S-learner regresses on both  $T$  and  $\mathbf{X}$ .

In the binary cases, the S-learner is usually considered as a good choice Künzel et al. (2019); Curth and van der Schaar (2021b) and have shown its performance. Although, as we will see in Section 5.1, its results are very sensitive to the base learner, particularly for random forests because it cannot capture the correct effect of the treatment variable.

Note that the S-learning approach may also suffer from the same regularization bias (Chernozhukov et al., 2018; Hahn et al., 2020) as the T-learning approach when estimating the counterfactual response model  $\hat{\mu}$ .

#### 4.2 Pseudo-outcome meta-learners

Despite the purpose of Proposition 4.1 for overcoming selection bias, it implies learning in low samples, which may harm them when  $\mathbf{S}_w$  becomes small for a certain  $w$ . This is all the more critical as the number  $K$  of treatments increases. An alternative (and usual) possibility for mitigating this bias is to consider some specific representations of the observed outcome  $Y_{\text{obs}}$ , called *pseudo-outcome*. These representations incorporate *nuisance components* that generally include valuable information such as the dependence between covariates  $\mathbf{X}$  and  $T$  (i.e. the GPS) and the occurrence of a particular treatment assignment. Further, regressing the pseudo-outcome produces a new *regularized* estimator that predicts the *right* treatment effect instead of predicting a biased effect while keeping the same sample size as  $Y_{\text{obs}}$ .

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**Algorithm 1** Pseudo-outcome meta-learning estimation
 

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**Input:** data  $(\mathbf{X}_i, T_i, Y_{\text{obs},i})$ , level  $t$ , model  $\bar{r}$ , Components  $\hat{r}, \hat{\mu}$ .

**if** Components not provided **then**

    Estimate  $\hat{r}$  by regressing  $T$  on  $\mathbf{X}$ .

    Estimate  $\hat{\mu}$  by T-learning or S-learning.

**end if**

$Z_{t,i} = \text{Transformation}(t, \mathbf{X}_i, T_i, Y_{\text{obs},i}, \hat{r}, \hat{\mu})$

Regress  $Z_t$  on  $\mathbf{X}$  using  $\bar{r}$ .

**Output:** Learned model  $\hat{r}$ .

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The algorithm 1 summarizes CATEs estimation using the previous meta-learners. The "Transformation" function stands for the pseudo-outcome transformation that has been applied to  $Y_{\text{obs}}$  in the following M-, DR- and X-learning approaches.

**M-learner with multiple treatments.** The M-learner (Athey and Imbens, 2016), where M- stands for the *modified* learned pseudo-outcome, is inspired from the Inverse Propensity Weighting (IPW) transformation. This representation is initially proposed by Horvitz and Thompson (1952) to the mean estimator, then presented by Rosenbaum (1987) as a form of model-based direct standardization for the inference of causal effects.

Let  $t \in \mathcal{T}$  be a treatment level, we define the *modified* pseudo-outcome  $Z_t^M$  in the multi-treatment regime using the IPW representation as

$$Z_t^M = \frac{\mathbf{1}\{T = t\}}{r(t, \mathbf{X})} Y_{\text{obs}} - \frac{\mathbf{1}\{T = t_1\}}{r(t_1, \mathbf{X})} Y_{\text{obs}}, \quad (2)$$

where  $r(t, \mathbf{x}) = \mathbb{P}(T = t \mid \mathbf{X} = \mathbf{x})$  is the GPS defined previously.

**Proposition 4.2.** *Under the assumptions (3.1)-(3.2)*

$$\mathbb{E}(Z_t^M \mid \mathbf{X} = \mathbf{x}) = \tau_t(\mathbf{x}).$$

Unfortunately, the M-learner is very sensitive to the estimation of the GPS and suffers from high variance, even when the propensity score is correctly specified or known and constant Curth and van der Schaar (2021a). Moreover, the *modified* pseudo-outcome can be null often, which also can lead to an over-fitting problem as the base-learner may try to predict zero instead of predicting  $\tau_t$ . Again, this becomes more critical as the number  $K$  of treatments increases as some values can be smaller than  $1/K$ .

**DR-learner with multiple treatments.** Requiring the consistency of the GPS estimator may be a hard condition to get a correct estimation of CATEs. The *Doubly Robust* (DR) method was suggested by Robins et al. (1994) then by Kennedy (2020) to overcome the problem of model's misspecification by estimating two components, the outcome model  $\mu_t$  and the GPS  $r$ , instead of relying on the correctness of one (and the only) parameter.

Let  $\bar{\mu}$  denote an arbitrary model of the outcome  $\mu$ , let  $\bar{r}$  denote also an arbitrary model of the GPS  $r$ , we assume that  $\bar{r}$  respects also Assumption (3.2). For  $t \in \mathcal{T}$ , we define *doubly-robust* pseudo-outcome  $Z_{\bar{\mu}, \bar{r}, t}^{DR}$  as

$$Z_{\bar{\mu}, \bar{r}, t}^{DR} = \frac{Y_{\text{obs}} - \bar{\mu}_T(\mathbf{X})}{\bar{r}(t, \mathbf{X})} \mathbf{1}\{T = t\} - \frac{Y_{\text{obs}} - \bar{\mu}_T(\mathbf{X})}{\bar{r}(t_1, \mathbf{X})} \mathbf{1}\{T = t_1\} + \bar{\mu}_t(\mathbf{X}) - \bar{\mu}_{t_1}(\mathbf{X}). \quad (3)$$

**Proposition 4.3.** *Let  $Z_{\bar{\mu}, \bar{r}, t}^{DR}$  be the Doubly-Robust pseudo-outcome defined in (3), then under the assumptions (3.1)-(3.2)*

$$\mathbb{E}(Z_{\bar{\mu}, \bar{r}, t}^{DR} \mid \mathbf{X} = \mathbf{x}) = \tau_t(\mathbf{X})$$

*if the outcome models or the propensity model is well-specified, i.e.  $\bar{\mu}_t(\mathbf{X}) = \mu_t(\mathbf{X})$  and  $\bar{\mu}_{t_1}(\mathbf{X}) = \mu_{t_1}(\mathbf{X})$  almost surely, or  $\bar{r}(\mathbf{X}) = r(\mathbf{X})$  almost surely.*

Therefore, the consistency of the DR-learner is achieved if at least one of the components (the propensity score model or outcome models) is well-specified. It also has the advantage of having a small asymptotic variance compared to the M-learner when the propensity score model is correct, as it will be shown in Appendix and Section 5.1.

**X-learner with multiple treatments.** The X-learner Künzel et al. (2019), also known as *Regression-Adjustment* (RA)-learning in a developed version by Curth and van der Schaar (2021a), has been proposed as an alternative to the T-learning in the case where one treatment group is over-represented. The idea consists in a *cross* procedure of estimation between observations  $Y_{\text{obs}}$  and outcome models when one of the treatments occurs.

In the multi-treatment regime, for  $t \neq t_1 \in \mathcal{T}$ , we define the *Regression-Adjustment* pseudo-outcome  $Z_t^X$  as

$$Z_t^X = \mathbf{1}\{T = t\}(Y_{\text{obs}} - \mu_{t_1}(\mathbf{X})) + \sum_{t' \neq t} \mathbf{1}\{T = t'\} \times (\mu_t(\mathbf{X}) - Y_{\text{obs}}) + \sum_{t' \neq t} \mathbf{1}\{T = t'\}(\mu_{t'}(\mathbf{X}) - \mu_{t_1}(\mathbf{X})). \quad (4)$$

**Proposition 4.4.** *Under the assumptions (3.1)-(3.2)*

$$\mathbb{E}(Z_t^X \mid \mathbf{X} = \mathbf{x}) = \tau_t(\mathbf{x}).$$

*Remark.* The X-learning approach provides also a new method for estimating the difference of Average Dose-Response Function (ARDF)  $\eta(t) = \mathbb{E}(Y(t) - Y(t_1))$ .

In opposition to the DR-learner, the pseudo-outcome  $Z_t^X$  incorporates only potential outcome models and does not imply the GPS  $r$ . Consequently, the X-learner is likely to have the smallest variance compared to other meta-learners when the GPS takes some extreme values (i.e. the overlap assumption (3.2) is not sufficiently respected). However, it requires the consistency of all components  $(\hat{\mu}_t)_{t \in \mathcal{T}}$  to estimate correctly the CATE.

### 4.3 Error estimation of pseudo-outcome meta-learners.

Seemingly, pseudo-outcome learners need to estimate components parameters on the same data  $\mathcal{D}_{\text{obs}}$ . Consequently, the pseudo-outcomes representations may, unfortunately, lead to higher variance (i.e. expected squared error) and poor performance in the way how these components intervene. In Appendix B, we analyze the estimation's error estimation of pseudo-outcome meta-learners and the variance upper bound of each meta-learner. It appears that:

- Without any surprises, the M-learner has the largest variance and its variance upper bound is constant.
- As the GPS is present in the denominator of the upper bounds of both M-learners and DR-learners. The variance is likely to be high when there is a lack of overlap in the propensity score, i.e.  $r_{\min} \rightarrow 0$ .
- The upper bounds of the X-learner and DR-learner depend on the potential outcomes model's estimation quality. One can expect that the more outcome models are precise, the lower the variance is.

**M-learner vs DR-learner.** If the potential outcome models are well-specified, the variance's upper bound is expected to be lower for the DR-learner. Controversially, suppose the outcome models are misspecified (but the propensity score is well-specified). In that case, there is no guarantee that the DR-learner would perform better than M-learner, and it may perform even worse, as we will see in some numerical results in Table 9 in Appendix D.

**X-learner vs DR-learner.** It is difficult to anticipate which meta-learner would perform better. This depends mainly on the expected squared error of  $\mu_w$ ,  $K$  and  $r_{\min}$ , whom, in some cases, make the X-learner having less error than the DR-learner, and the opposite in other cases. Still, numerical results in Appendix D (Tables 4, 9, 11 and 13) show that the X-learner outperforms the DR-learner when the nuisance components are exact.

### 4.4 R-learning approach

Being processing in two steps, the R-learner was proposed by Nie and Wager (2020) to estimate heterogeneous treatment effects in the binary setting: Estimating nuisance components then estimating the effects by minimizing a loss function (model-fitting). The R-learner is based mainly on the Robinson (1988) decomposition to provide a flexible estimator avoiding regularization bias, with strong convergence rates. Principally, the R-learner achieves approximately asymptotic error rates as an "oracle" learner knowing the nuisance parameters perfectly.

The following proposition, which is slightly different from the work of Kaddour et al. (2021), aims to generalize the Robinson (1988) representation in the multi-treatment setting without assuming Product Decomposition of  $Y_{\text{obs}}$ .

**Proposition 4.5.** *In the multi-treatment regime, let  $\epsilon$  be the outcome model error*

$$\epsilon = Y_{\text{obs}} - \sum_{t \in \mathcal{T}} \mathbf{1}\{T = t\} \mu_t(\mathbf{X}) = Y_{\text{obs}} - \mu_T(\mathbf{X})$$

Then  $\epsilon$  satisfies  $\mathbb{E}(\epsilon | T, \mathbf{X}) = 0$  (Neyman Orthogonality) and the decomposition

$$\epsilon = Y_{\text{obs}} - m(\mathbf{X}) - \sum_{t \neq t_1 \in \mathcal{T}} (\mathbf{1}\{T = t\} - r(t, \mathbf{X}))\tau_t(\mathbf{X})$$

where  $m(\mathbf{x}) = \mathbb{E}(Y_{\text{obs}} | \mathbf{X} = \mathbf{x})$  is the observed outcome model,  $r(t, \mathbf{x}) = \mathbb{P}(T = t | \mathbf{X} = \mathbf{x})$  is the GPS.

As described in the original paper of Nie and Wager (2020), the main interest of the previous decomposition relies on forming a pseudo-outcome error, implying only the regression of observed quantities on  $\mathbf{X}$  (i.e. the observed outcome model  $m$  and the GPS  $r$ ), that isolates CATEs  $\tau_t$  for all  $t \neq t_1$ . The generalized Robinson decomposition is relevant for two reasons. Firstly, setting up an error to minimize that allows us to target CATEs models  $\tau_t$  directly Kaddour et al. (2021). Secondly, requiring the observed outcome model is less restrictive than requiring potential outcome models  $\mu$ , as in the DR- and X- pseudo-outcomes.

In the multi-treatment regime, considering the mean squared error of  $\epsilon$  as loss function and minimizing it implies estimating simultaneously  $K - 1$  models  $\{\tau_t\}_{t \neq t_1 \in \mathcal{T}}$  such that

$$\begin{aligned} \{\hat{\tau}_t^{(R)}\}_{t \neq t_1 \in \mathcal{T}} = \arg \min_{\{\tau_t\}_{t \neq t_1 \in \mathcal{T}} \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n \left[ (Y_{\text{obs},i} - \hat{m}(\mathbf{X}_i)) - \sum_{t \neq t_1 \in \mathcal{T}} (\mathbf{1}\{T_i = t\} - \hat{r}(t, \mathbf{X}_i))\tau_t(\mathbf{X}_i) \right]^2, \end{aligned} \quad (5)$$

where  $\hat{m}$  (respectively,  $\hat{r}$ ) is an estimator of  $m$  (respectively,  $r$ ) and  $\mathcal{F}$  is the space of candidate models  $[\{\tau_t\}_{t \neq t_1}]$ .

Still, the major difficulty with our R-learning approach in the multi-treatment regime comes from the fact that Problem (5) cannot be written similarly as *weighted* supervised learning problem with a specific pseudo-outcome. Therefore, only parametric families  $\mathcal{F}$  can be considered in the multi-treatment regime.

**Proposition 4.6.** *Let assume that  $\tau_t$  belongs to the family of linear regression models, then Problem (5) admits at least a solution, given by Ordinary Least Squares estimator.*

**Proposition 4.7.** *Let assume that  $\tau_t$  belongs to the Reproducing Kernel Hilbert Space (RKHS) with a reproducing kernel  $k$  and hyperparameters  $(\sigma^2, \boldsymbol{\theta})$ , then Problem (5) admits at least a solution, whose regression coefficients are given by Ordinary Least Squares estimator and optimal hyperparameters are solved numerically.*

We note that the kernel regression method is heavy to solve (cost of  $O(n^3 K^3)$  at each iteration). Thus, we do not present its results in Section 5 and limit ourselves only to R-learners derived from linear regression.

## 5 Experiments and numerical results

We remind that our main goal is to build models able to estimate CATEs as precise as possible for the in-sample counterfactual prediction (i.e. for the same observed covariates  $\mathbf{X}$  but different treatment level  $T$ ) but also, ideally, for out-sample counterfactual prediction for decision-making purposes. However, as mentioned in Section 2, even the task of in-sample prediction is still difficult as realizations of the true CATE  $\tau_t$  are not observable, thus, accurate performance estimation is difficult. Therefore, training our models on sample  $\mathcal{D}_{\text{obs}}$  and predicting on the same sample is quite different from *standard* in-sample prediction and seems somehow as an out-sample prediction if compared to *classical* supervised regression problem.

**Metric.** In the examples where the potential outcome functions and/or CATEs are *a priori* known (see subsections 5.1 and 5.2), the error in estimation is given by **mPEHE**, the mean of the Precision in Estimation of Heterogeneous Effect (PEHE) (Hill, 2011; Shalit et al., 2017) defined as the mean squared error in the estimation of the treatment effect  $\hat{\tau}_t$ , over all possible treatment levels  $t \in \mathcal{T} \setminus \{t_1\}$ :

$$\mathbf{mPEHE} = \frac{1}{K-1} \sum_{t \neq t_1} PEHE(\hat{\tau}_t),$$

where  $PEHE(\hat{\tau}_t) = \sqrt{\frac{1}{n} \sum_{i=1}^n (\hat{\tau}_t(\mathbf{X}_i) - \tau_t(\mathbf{X}_i))^2}$ .

This metric will be used to compare and identify conditions (sample size  $n$ , number of possible treatments  $K$ , the correctness of nuisance parameters and base-learners) under which we can precisely estimate CATEs. We do not consider here model-fitting of base-learners. More specifically, all hyperparameters (e.g. number of trees, depth etc.) are fixed to their default values during all experiments.

### 5.1 Synthetic datasets: analytical functions in randomized and non-randomized studies

In this subsection, we begin by empirically evaluating the performance of our meta-learners when the treatment  $T$  is taking  $K = 10$  possible values in  $[0, 1]$  in a RCT setting where the outcome is a linear model and satisfies:

$$Y(t) | X \sim \mathcal{N}((1+t)X, \sigma^2), X \sim \mathcal{U}[0, 1].$$

then, we evaluate meta-learners on the hazard rate outcome:

$$Y(t) | \mathbf{X} \sim \mathcal{N}(t + \|\mathbf{X}\| \exp(-t\|\mathbf{X}\|), \sigma^2),$$

for  $\mathbf{X} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_5)$  in a non-randomized setting.

To simulate observational data, instead of removing some rows, we propose to create a bias in the data by selecting preferentially only observations with specific characteristics (see subsection C.1.2 in Appendix). This strategy comes in line with the findings and recommendations of Curth et al. (2021) about the creation of biased sub-sample and evaluation of CATE estimators.

The GPS is estimated using gradient boosting models (XGBoost), and the outcome models  $\mu_w$  are either estimated by the T-learning (left value in the table for the DR and X-learners) or S-learning approaches (right value in the table). In the following tables and Appendix D, the bold font is to indicate the best meta-learner (row) per base-learner (column), and parenthesis indicates with which estimation of  $\mu_w$  we get better results.

Table 1: **mPEHE** for XGBoost and RandomForest; linear outcome in RCT setting with  $n = 2000$  units.

| Meta-learner | XGBoost       | RandomForest  |
|--------------|---------------|---------------|
| T-Learner    | 0.061         | <b>0.037</b>  |
| S-Learner    | <b>0.029</b>  | 0.040         |
| M-Learner    | 1.23          | 1.15          |
| DR-Learner   | 0.063 (0.063) | (0.060) 0.060 |
| X-Learner    | 0.059 (0.030) | (0.041) 0.079 |
| RLin-Learner | 0.122         | 0.112         |

In Tables 1 and 2, we find that, as expected, the M-learner predicts poorly. The T-learner gives better predictions for Random Forest, whereas the S-learner gives better results for XGBoost. Regularizing T-learner (*RegT-Learner*) increases its performances. The X- and DR-learners improve the predictions of the S-learner for XGBoost, but this improvement is not always observable for Random Forests. Surprisingly, the R-learner outperforms when combined with Random Forests for the Hazard rate model.

Table 2: **mPEHE** for XGBoost and RandomForest. Hazard rate function in observational setting with  $n = 10000$  units.

| Meta-learner        | XGboost              | RandomForest  |
|---------------------|----------------------|---------------|
| T-Learner           | 0.184                | 0.251         |
| <i>RegT-Learner</i> | 0.158                | 0.253         |
| S-Learner           | 0.166                | 0.269         |
| M-Learner           | 1.56                 | 1.55          |
| DR-Learner          | (0.151) 0.171        | (0.275) 0.288 |
| X-Learner           | <b>(0.149)</b> 0.162 | (0.270) 0.286 |
| RLin-Learner        | 0.235                | <b>0.178</b>  |

Despite these satisfying results, we highlight the problem of over-fitting of gradient boosting models and Random Forest by comparing them with the linear model in Appendix D. This problem should be taken seriously while estimating CATEs.

Finally, on the one hand, when  $K$  increases, the R-learner becomes more effective for CATEs prediction, but the performance of the T-learner becomes compromised, with a slight impact on other learners. We recommend, therefore, the estimated potential outcome model by the S-learner when  $K \geq 10$  for pseudo-outcome meta-learners. On the other



hand, having a large sample size  $n$  improves the quality of the meta-learner’s estimation (except for the M-learner, probably due to estimation of  $r$ ). To conclude, when applying gradient boosting machine as base-learner, two-step meta-learners are robust. In particular, the X-learner improves the quality of one-step meta-learners, and when it does not, the differences are very small.

## 5.2 Semi-synthetic dataset: simulating Enhanced Geothermal System with physics-based models

**Motivation and description.** The difficulty in evaluating a causal model’s performance in real-world applications motivates the need to create a semi-synthetic dataset. In this subsection, we consider a multistage fracturing Enhanced Geothermal System (EGS). We assume that the heat extraction performance satisfies the following physical model:  $Q_{well}(\ell_L) = Q_{fracture} \times \ell_L/d \times \eta_d$ , where  $Q_{fracture}$  is the *unknown* heat extraction performance from a single fracture, that can be generated using eight parameters including reservoir characteristics and fracture design,  $\ell_L$  is the Lateral Length of the well,  $d$  is the average spacing between two fractures and  $\eta_d$  is the stage efficiency penalizing the individual contribution when fractures are close to each other. This model respects the unconfoundedness assumption (3.1), and we can control all its variables in the simulations.

A randomized series of numerical experiments using a numerical emulator has been conducted to simulate the heat performance from a single fracture (i.e.  $Q_{fracture}$ ), leading to an initial full factorial design of experiments dataset of 16200 observations covering all possible scenarios of a fracture in a reservoir. The final dataset, containing around one million observations, is created by extrapolating the heat performance of each case to all well’s lateral lengths, fracture spacing and the efficiency coefficient (See Appendix E for more details.)

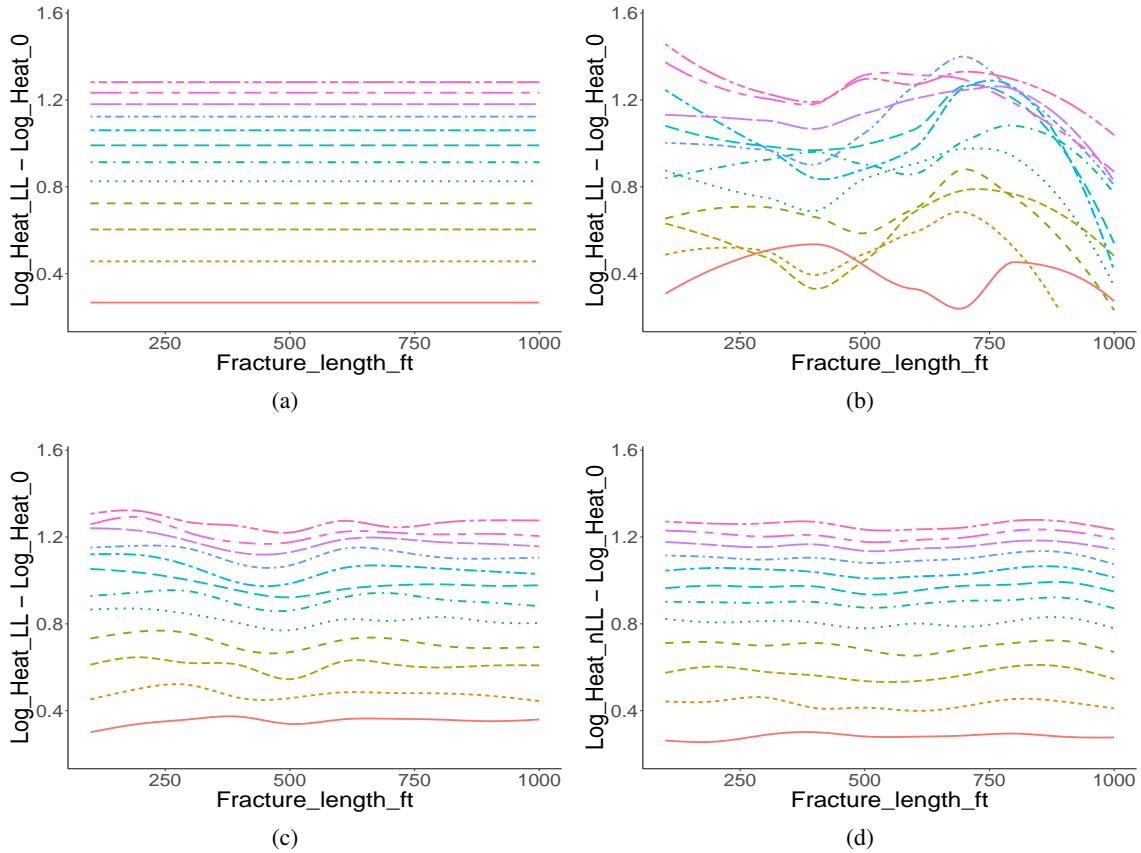


Figure 1: CATEs estimation on semi-synthetic dataset. Each line represents  $\tau_j$  for  $j = 1, \dots, K$ . (a): The ground truth model; (b): A biased estimation of CATEs by regressing on  $\text{Fracture\_length\_ft}$ ; (c): T-learner estimation; (d): X-learner estimation.

The creation of this semi-synthetic dataset is also one of the main contributions of this paper as it serves as a *ground truth model* for validating causal inference methods.

**Estimating Heterogeneous Treatment Effects on a non-randomized biased dataset.** We consider the Lateral Length as treatment  $T$  with  $K = 13$  possible values and the covariates  $\mathbf{X} \in \mathbb{R}^{11}$  are the remaining variables. We also consider a logarithmic transformation of the heat performance for a meaningful **mPEHE**, and we normalize the treatment  $T$ . Following the *preferential selection*, we sample  $n = 10000$  units such that wells with high lateral length are likely to have larger fractures and vice versa. The GPS is estimated using gradient boosting models. Table 3 resumes the **mPEHE** for different meta-learners. Most findings as subsection 5.1 remain valid: XGBoost model is generally a better choice than Random Forests (except for T-learning); The X-learner, followed by DR-learner, outperforms all other learners.

 Table 3: **mPEHE** for XGBoost and RandomForest

| Meta-learner | XGBoost                | RandomForest  |
|--------------|------------------------|---------------|
| T-learner    | 0.167                  | 0.154         |
| RegT-Learner | 0.153                  | <b>0.153</b>  |
| S-learner    | 0.101                  | 0.216         |
| M-learner    | 1.05                   | 0.907         |
| DR-learner   | 0.146 (0.100)          | (0.162) 0.199 |
| X-learner    | 0.140 ( <b>0.095</b> ) | (0.175) 0.209 |
| RLin-learner | 0.336                  | 0.338         |

Finally, Figure 1 shows the ground truth model, what would one obtain by regressing only on fracture length (correlation) and T-, X-learner’s estimation. It demonstrates the ability of meta-learners, in particular the X-learner, to rebuild the ground truth.

## 6 Discussion and Future work

In this paper, we investigated heterogeneous treatment effects estimation under multi-valued treatment. In addition to standard plug-in meta-learners, we considered new representations to build pseudo-outcome meta-learners, and we analyzed this class’s error variance. We also proposed the generalized Robinson decomposition to build the R-learner. Through synthetic datasets and a semi-synthetic industrial dataset, we assessed the performances of different meta-learners in a non-randomized case where some covariates are confounded with the treatment. We showed, in particular, the ability of the X-learner to reconstruct the ground truth model. We also highlighted how the choice of base-learner can affect the quality of CATEs estimation. Precisely, it is recommended to choose gradient boosting machines than random forests. Finally, we may agree that our R-learner should be reviewed. This will allow us also to apply non-parametric methods to it and compare it with other ML methods. The next step would be to extend this approach to continuous treatments.

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## A Proofs of propositions 4.1 - 4.6.

### Proof of Proposition 4.1: Regularizing T-learner to selection bias.

*Proof.* This proof is similar to the proof of equation (5) in supplementary of Curth and van der Schaar (2021a). Let  $p(\mathbf{x})$  denotes the PDF of  $\mathbf{X}$  under  $\mathbb{P}(\cdot)$ ,  $p(\mathbf{x} | T = t)$  the PDF of the conditional law of  $\mathbf{X}|T$  and  $R_t = \int (\hat{\mu}_t(\mathbf{x}) - \mu_t(\mathbf{x}))^2 p(\mathbf{x} | T = t) d\mathbf{x}$

$$\begin{aligned}
 \mathbb{E}_{\mathbf{X} \sim \mathbb{P}(\cdot)} [(\hat{\mu}_t(\mathbf{X}) - \mu_t(\mathbf{X}))^2] &= \int (\hat{\mu}_t(\mathbf{x}) - \mu_t(\mathbf{x}))^2 p(\mathbf{x}) d\mathbf{x} \\
 &= \mathbb{P}(T = t) \int (\hat{\mu}_t(\mathbf{x}) - \mu_t(\mathbf{x}))^2 p(\mathbf{x} | T = t) d\mathbf{x} + \sum_{t' \neq t} \mathbb{P}(T = t') \int (\hat{\mu}_t(\mathbf{x}) - \mu_t(\mathbf{x}))^2 p(\mathbf{x} | T = t') d\mathbf{x} \\
 &= \mathbb{P}(T = t) R_t + \sum_{t' \neq t} \mathbb{P}(T = t') \int (\hat{\mu}_t(\mathbf{x}) - \mu_t(\mathbf{x}))^2 \frac{p(\mathbf{x} | T = t')}{p(\mathbf{x} | T = t)} p(\mathbf{x} | T = t) d\mathbf{x} \\
 &= \mathbb{P}(T = t) R_t + \sum_{t' \neq t} \mathbb{P}(T = t') \int (\hat{\mu}_t(\mathbf{x}) - \mu_t(\mathbf{x}))^2 \frac{\frac{\mathbb{P}(T=t'|\mathbf{x})p(\mathbf{x})}{\mathbb{P}(T=t|\mathbf{x})p(\mathbf{x})}}{\frac{\mathbb{P}(T=t)}{\mathbb{P}(T=t)}} p(\mathbf{x} | T = t) d\mathbf{x} \quad (\text{Bayes rule}) \\
 &= \mathbb{P}(T = t) R_t + \mathbb{P}(T = t) \sum_{t' \neq t} \int (\hat{\mu}_t(\mathbf{x}) - \mu_t(\mathbf{x}))^2 \frac{\mathbb{P}(T = t' | \mathbf{x})}{\mathbb{P}(T = t | \mathbf{x})} p(\mathbf{x} | T = t) d\mathbf{x} \quad (6) \\
 &= \mathbb{P}(T = t) R_t + \mathbb{P}(T = t) \int (\hat{\mu}_t(\mathbf{x}) - \mu_t(\mathbf{x}))^2 \frac{\sum_{t' \neq t} \mathbb{P}(T = t' | \mathbf{x})}{\mathbb{P}(T = t | \mathbf{x})} p(\mathbf{x} | T = t) d\mathbf{x} \\
 &= \mathbb{P}(T = t) R_t + \mathbb{P}(T = t) \int \frac{1 - r(t, \mathbf{x})}{r(t, \mathbf{x})} (\hat{\mu}_t(\mathbf{x}) - \mu_t(\mathbf{x}))^2 p(\mathbf{x} | T = t) d\mathbf{x} \\
 &= \mathbb{P}(T = t) \int \left(1 + \frac{1 - r(t, \mathbf{x})}{r(t, \mathbf{x})}\right) (\hat{\mu}_t(\mathbf{x}) - \mu_t(\mathbf{x}))^2 p(\mathbf{x} | T = t) d\mathbf{x} \\
 &= \mathbb{E}_{\mathbf{X} \sim \mathbb{P}(\cdot | T=t)} \left[ \frac{\mathbb{P}(T = t)}{r(t, \mathbf{X})} (\hat{\mu}_t(\mathbf{X}) - \mu_t(\mathbf{X}))^2 \right].
 \end{aligned}$$

□

### Proof of Proposition 4.2: Consistency of the M-learner.

*Proof.* We consider  $Y_t^M$  the modified IPW representation of  $Y_{\text{obs}}$  in such way that  $Z_t^M = Y_t^M - Y_{t_1}^M$ . By noticing that  $\mathbf{1}\{T = t\}Y_{\text{obs}} = \mathbf{1}\{T = t\}Y(t)$ , we have for  $\mathbf{x} \in \mathcal{X}$ :

$$\begin{aligned}
 \mathbb{E}(Y_t^M | \mathbf{X} = \mathbf{x}) &= \mathbb{E} \left[ \frac{\mathbf{1}\{T = t\}}{r(t, \mathbf{X})} Y_{\text{obs}} | \mathbf{X} = \mathbf{x} \right] \\
 &= \frac{1}{r(t, \mathbf{x})} \mathbb{E}[\mathbf{1}\{T = t\}Y(t) | \mathbf{X} = \mathbf{x}] \\
 &= \frac{1}{r(t, \mathbf{x})} \mathbb{E}[\mathbf{1}\{T = t\} | \mathbf{X} = \mathbf{x}] \mathbb{E}[Y(t) | \mathbf{X} = \mathbf{x}] \quad (\text{by Assumption 3.1}) \\
 &= \mathbb{E}(Y(t) | \mathbf{X} = \mathbf{x}) = \mu_t(\mathbf{x})
 \end{aligned} \quad (7)$$

Thus  $\mathbb{E}(Z_t^M | \mathbf{X} = \mathbf{x}) = \mu_t(\mathbf{x}) - \mu_{t_1}(\mathbf{x})$  and we get the desired result. □

### Proof of Proposition 4.3: Consistency of the DR-learner.

*Proof.* Let  $\bar{\mu}$  denote an arbitrary model of the outcome  $\mu$ , let  $\bar{r}$  denote also an arbitrary model of the GPS  $r$  satisfying the overlap assumption 3.2. Similarly to the previous proof, we consider  $Y_t^{DR}$  the AIPW representation of  $Y_{\text{obs}}$  such

that  $Z_{\bar{\mu}, \bar{r}, t}^{DR} = Y_{\bar{\mu}, \bar{r}, t}^{DR} - Y_{\bar{\mu}, \bar{r}, t_1}^{DR}$  and we show that

$$\begin{aligned}
 \mathbb{E}(Y_{\bar{\mu}, \bar{r}, t}^{DR} | \mathbf{X} = \mathbf{x}) &= \mathbb{E} \left[ \frac{Y_{\text{obs}} - \bar{\mu}_T(\mathbf{X})}{\bar{r}(t, \mathbf{X})} \mathbf{1}\{T = t\} + \bar{\mu}_t(\mathbf{X}) \mid \mathbf{X} = \mathbf{x} \right] \\
 &= \mathbb{E} \left[ \frac{Y(t) - \bar{\mu}_t(\mathbf{X})}{\bar{r}(t, \mathbf{X})} \mathbf{1}\{T = t\} + \bar{\mu}_t(\mathbf{X}) \mid \mathbf{X} = \mathbf{x} \right] \\
 &= \mathbb{E}[Y(t) \mid \mathbf{X} = \mathbf{x}] + \mathbb{E} \left[ \frac{Y(t) - \bar{\mu}_t(\mathbf{X})}{\bar{r}(t, \mathbf{X})} \mathbf{1}\{T = t\} - Y(t) + \bar{\mu}_t(\mathbf{X}) \mid \mathbf{X} = \mathbf{x} \right] \\
 &= \mu_t(\mathbf{x}) + \eta_t(\mathbf{x}),
 \end{aligned} \tag{8}$$

with  $\eta_t(\mathbf{x}) = \mathbb{E} \left[ \frac{\mathbf{1}\{T=t\} - \bar{r}(t, \mathbf{X})}{\bar{r}(t, \mathbf{X})} (Y(t) - \bar{\mu}_t(\mathbf{X})) \mid \mathbf{X} = \mathbf{x} \right]$ .

We show that the second term  $\eta_t$  is null under the double robustness of the model, that is, if one of the nuisance components is consistent.

- If the propensity model  $\bar{r}$  is correctly specified (i.e.  $\bar{r}(T, \mathbf{X}) = r(T, \mathbf{X}) = \mathbb{P}(T \mid \mathbf{X})$  *almost surely*) but the outcome model is misspecified, we would have

$$\begin{aligned}
 \eta_t(\mathbf{x}) &= \mathbb{E} \left[ \frac{\mathbf{1}\{T = t\} - \bar{r}(t, \mathbf{X})}{\bar{r}(t, \mathbf{X})} (Y(t) - \bar{\mu}_t(\mathbf{X})) \mid \mathbf{X} = \mathbf{x} \right] \\
 &= \mathbb{E} \left[ \mathbb{E} \left[ \frac{\mathbf{1}\{T = t\} - \bar{r}(t, \mathbf{X})}{\bar{r}(t, \mathbf{X})} (Y(t) - \bar{\mu}_t(\mathbf{X})) \mid Y(t), \mathbf{X} \right] \mid \mathbf{X} = \mathbf{x} \right] \\
 &= \mathbb{E} \left[ (Y(t) - \bar{\mu}_t(\mathbf{X})) \mathbb{E} \left[ \frac{\mathbf{1}\{T = t\} - \bar{r}(t, \mathbf{X})}{\bar{r}(t, \mathbf{X})} \mid Y(t), \mathbf{X} \right] \mid \mathbf{X} = \mathbf{x} \right] \\
 &= \mathbb{E} \left[ (Y(t) - \bar{\mu}_t(\mathbf{X})) \frac{\mathbb{E}[\mathbf{1}\{T = t\} \mid Y(t), \mathbf{X}] - \bar{r}(t, \mathbf{X})}{\bar{r}(t, \mathbf{X})} \mid \mathbf{X} = \mathbf{x} \right] \\
 &= \mathbb{E} \left[ (Y(t) - \bar{\mu}_t(\mathbf{X})) \frac{\mathbb{E}[\mathbf{1}\{T = t\} \mid \mathbf{X}] - \bar{r}(t, \mathbf{X})}{\bar{r}(t, \mathbf{X})} \mid \mathbf{X} = \mathbf{x} \right] \quad (\text{by Assumption 3.1}) \\
 &= \mathbb{E} \left[ (Y(t) - \bar{\mu}_t(\mathbf{X})) \frac{r(t, \mathbf{X}) - \bar{r}(t, \mathbf{X})}{\bar{r}(t, \mathbf{X})} \mid \mathbf{X} = \mathbf{x} \right] = 0,
 \end{aligned} \tag{9}$$

where the last line holds by the definition the Generalize Propensity Score  $r(T, X)$ .

- If the propensity model is misspecified but the outcome model is correctly specified (i.e.  $\bar{\mu} = \mu = \mathbb{E}(Y_{\text{obs}} \mid T, \mathbf{X})$  *almost surely*), we would have

$$\begin{aligned}
 \eta_t(\mathbf{x}) &= \mathbb{E} \left[ \frac{\mathbf{1}\{T = t\} - \bar{r}(T, \mathbf{X})}{\bar{r}(T, \mathbf{X})} (Y(t) - \mathbb{E}(Y_{\text{obs}} \mid T = t, \mathbf{X})) \mid \mathbf{X} = \mathbf{x} \right] \\
 &= \mathbb{E} \left[ \mathbb{E} \left[ \frac{\mathbf{1}\{T = t\} - \bar{r}(T, \mathbf{X})}{\bar{r}(T, \mathbf{X})} (Y(t) - \mathbb{E}(Y_{\text{obs}} \mid T = t, \mathbf{X})) \mid T, \mathbf{X} \right] \mid \mathbf{X} = \mathbf{x} \right] \\
 &= \mathbb{E} \left[ \frac{\mathbf{1}\{T = t\} - \bar{r}(t, \mathbf{X})}{\bar{r}(t, \mathbf{X})} (\mathbb{E}[Y(t) \mid T, \mathbf{X}] - \mathbb{E}(Y_{\text{obs}} \mid T = t, \mathbf{X})) \mid \mathbf{X} = \mathbf{x} \right] \\
 &= \mathbb{E} \left[ \frac{\mathbf{1}\{T = t\} - \bar{r}(t, \mathbf{X})}{\bar{r}(t, \mathbf{X})} \mathbb{E}([Y_{\text{obs}} \mid T = t, \mathbf{X}] - \mathbb{E}[Y_{\text{obs}} \mid T = t, \mathbf{X}]) \mid \mathbf{X} = \mathbf{x} \right] \\
 &= 0.
 \end{aligned} \tag{10}$$

Note that assuming  $\bar{\mu}_t = \mu_t = \mathbb{E}(Y_{\text{obs}} \mid T = t, \mathbf{X})$  is sufficient to prove that  $\eta(\mathbf{x}) = 0$ . The result holds similarly for  $Y_{\bar{\mu}, \bar{r}, t_1}^{DR}$ . Therefore, the consistency of DR-learner is achieved if the propensity score is well-specified or if the potential outcome model is well-specified (at least for  $t$  and  $t_1$ ).  $\square$

**Proof of Proposition 4.4: Consistency of the X-learner.**

*Proof.* By direct calculations, we show that

$$\begin{aligned}
 \mathbb{E}(Z_t^X \mid \mathbf{X} = \mathbf{x}) &= \mathbb{E}[\mathbf{1}\{T = t\}Y(t) \mid \mathbf{X} = \mathbf{x}] - r(t, \mathbf{x})\mu_{t_1}(\mathbf{x}) + \sum_{t' \neq t} r(t', \mathbf{x}) \left( \mu_t(\mathbf{x}) - \mathbb{E}[\mathbf{1}\{T = t'\}Y(t') \mid \mathbf{X} = \mathbf{x}] \right) \\
 &\quad + \sum_{t' \neq t} r(t', \mathbf{x}) (\mu_{t'}(\mathbf{x}) - \mu_{t_1}(\mathbf{x})) \\
 &= r(t, \mathbf{x})\mu_t(\mathbf{x}) - r(t, \mathbf{x})\mu_{t_1}(\mathbf{x}) + \sum_{t' \neq t} (r(t', \mathbf{x})\mu_t(\mathbf{x}) - r(t', \mathbf{x})\mu_{t'}(\mathbf{x})) \\
 &\quad + \sum_{t' \neq t} r(t', \mathbf{x}) (\mu_{t'}(\mathbf{x}) - \mu_{t_1}(\mathbf{x})) \quad (\text{by Assumption 3.1}) \\
 &= r(t, \mathbf{x})\mu_t(\mathbf{x}) - r(t, \mathbf{x})\mu_{t_1}(\mathbf{x}) + \sum_{t' \neq t} r(t', \mathbf{x})\mu_t(\mathbf{x}) - \sum_{t' \neq t} r(t', \mathbf{x})\mu_{t_1}(\mathbf{x}) \\
 &= (\mu_t(\mathbf{x}) - \mu_{t_1}(\mathbf{x})) \left( r(t, \mathbf{x}) + \sum_{t' \neq t} r(t', \mathbf{x}) \right) \\
 &= \mu_t(\mathbf{x}) - \mu_{t_1}(\mathbf{x}) = \tau_t(\mathbf{x}).
 \end{aligned}$$

□

**Proof of Proposition 4.5: the generalized Robinson decomposition.**

*Proof.* We show first the Neyman orthogonality propriety, i.e.  $\mathbb{E}(\epsilon \mid T, \mathbf{X}) = 0$ . Indeed, we have

$$\begin{aligned}
 \mathbb{E}[\epsilon \mid T = t, \mathbf{X} = \mathbf{x}] &= \mathbb{E}[Y_{\text{obs}} - \mu_T(\mathbf{X}) \mid T = t, \mathbf{X} = \mathbf{x}] \\
 &= \mathbb{E}[Y(t) - \mu_T(\mathbf{X}) \mid T = t, \mathbf{X} = \mathbf{x}] \\
 &= \mu_t(\mathbf{x}) - \mu_t(\mathbf{x}) = 0.
 \end{aligned} \tag{11}$$

Thus, the observed outcome model satisfies

$$\begin{aligned}
 \mathbb{E}(Y_{\text{obs}} \mid \mathbf{X} = \mathbf{x}) &= \mathbb{E}\left[\epsilon + \sum_{t \in \mathcal{T}} \mathbf{1}\{T = t\}\mu_t(\mathbf{X}) \mid \mathbf{X} = \mathbf{x}\right] \\
 &= \mathbb{E}[\mathbb{E}[\epsilon \mid T, \mathbf{X}] \mid \mathbf{X} = \mathbf{x}] + \sum_{t \in \mathcal{T}} \mathbb{E}[\mathbf{1}\{T = t\} \mid \mathbf{X} = \mathbf{x}] \mu_t(\mathbf{x}) \\
 &= \sum_{t \in \mathcal{T}} \mu_t(\mathbf{x}) r(t, \mathbf{x}) = \mu_{t_1}(\mathbf{x}) r(t_1, \mathbf{x}) + \sum_{t \neq t_1 \in \mathcal{T}} \mu_t(\mathbf{x}) r(t, \mathbf{x}) \\
 &= \mu_{t_1}(\mathbf{x}) \left[ 1 - \sum_{t \neq t_1 \in \mathcal{T}} r(t, \mathbf{x}) \right] + \sum_{t \neq t_1 \in \mathcal{T}} \mu_t(\mathbf{x}) r(t, \mathbf{x}) \\
 &= \mu_{t_1}(\mathbf{x}) + \sum_{t \neq t_1 \in \mathcal{T}} r(t, \mathbf{x}) [\mu_t(\mathbf{x}) - \mu_{t_1}(\mathbf{x})] \\
 &= \mu_{t_1}(\mathbf{x}) + \sum_{t \neq t_1 \in \mathcal{T}} r(t, \mathbf{x}) \tau_t(\mathbf{x}) = m(\mathbf{X}).
 \end{aligned} \tag{12}$$





where

$$\begin{aligned}\mathbf{a}_j &= \frac{1}{n} \mathbf{F}^\top \mathbf{D}_{\bar{T}_j} \bar{Y} \in \mathbb{R}^p, \\ \mathbf{B}_j &= \frac{1}{n} \mathbf{F}^\top \mathbf{D}_{\bar{T}_j}^2 \mathbf{F} \in \mathbb{R}^{p \times p}, \\ \mathbf{C}_{ij} &= \frac{1}{n} \mathbf{F}^\top \mathbf{D}_{\bar{T}_i} \mathbf{D}_{\bar{T}_j} \mathbf{F} \in \mathbb{R}^{p \times p}.\end{aligned}$$

Let  $\boldsymbol{\beta} = (\boldsymbol{\beta}_2^\top, \dots, \boldsymbol{\beta}_K^\top)^\top \in \mathbb{R}^{(K-1)p}$  and consider the block matrix  $\mathbf{A}$  defined as.

$$\mathbf{A} = \begin{bmatrix} \mathbf{B}_2 & \mathbf{C}_{23} & \cdots & \mathbf{C}_{2K} \\ \mathbf{C}_{32} & \mathbf{B}_3 & \cdots & \mathbf{C}_{3K} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}_{K2} & \mathbf{C}_{K3} & \cdots & \mathbf{B}_K \end{bmatrix}. \quad (16)$$

The matrix  $\mathbf{A}$  is real symmetric and satisfies

$$\begin{aligned}\boldsymbol{\beta}^\top \mathbf{A} \boldsymbol{\beta} &= \sum_{2 \leq k, l \leq K} \boldsymbol{\beta}_k^\top \mathbf{F}^\top \mathbf{D}_{\bar{T}_k} \mathbf{D}_{\bar{T}_l} \mathbf{F} \boldsymbol{\beta}_l \\ &= \left\| \sum_{k=2}^K \mathbf{D}_{\bar{T}_k} \mathbf{F} \boldsymbol{\beta}_k \right\|^2 \geq 0.\end{aligned}$$

This result shows that  $\mathbf{A}$  is positive semi-definite, all its eigenvalues are positive and also proves the existence of a minimizer  $\hat{\boldsymbol{\beta}}$  to the loss function  $\mathcal{L}$ .

The optimal solution  $\hat{\boldsymbol{\beta}}$  to the problem (5) can be given by

$$\hat{\boldsymbol{\beta}} = \mathbf{A}^+ \mathbf{a}, \quad (17)$$

where  $\mathbf{A}^+$  is the Moore–Penrose inverse of  $\mathbf{A}$  and  $\mathbf{a} = (\mathbf{a}_2^\top, \dots, \mathbf{a}_K^\top)^\top$ .

*Remark.* If  $\mathbf{D}_{\bar{T}_k} \boldsymbol{\beta}_k \notin \text{Im}(\mathbf{F})^\perp$  for all  $k \in \{2, \dots, K\}$ , then  $\sum_{k=2}^K \mathbf{D}_{\bar{T}_k} \boldsymbol{\beta}_k \notin \text{Im}(\mathbf{F})^\perp = \text{Ker}(\mathbf{F}^\top)$  which is sufficient to prove that  $\mathbf{A}$  is positive definite. In this case, the system in (62) admits a unique solution such that

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\boldsymbol{\beta}}_2 \\ \hat{\boldsymbol{\beta}}_3 \\ \vdots \\ \hat{\boldsymbol{\beta}}_K \end{bmatrix} = \begin{bmatrix} \mathbf{B}_2 & \mathbf{C}_{23} & \cdots & \mathbf{C}_{2K} \\ \mathbf{C}_{32} & \mathbf{B}_3 & \cdots & \mathbf{C}_{3K} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}_{K2} & \mathbf{C}_{K3} & \cdots & \mathbf{B}_K \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{a}_2 \\ \mathbf{a}_3 \\ \vdots \\ \mathbf{a}_K \end{bmatrix}. \quad (18)$$

□

**R-learning with kernel regression family.** In this proof, we introduce the Kernel regression framework as developed by Schölkopf and Smola (2001). This framework is based on considering the Hilbert space  $\mathcal{H}$  defined as the closure of the linear span of the functions  $f : \mathbf{x} \mapsto \mathbf{k}(\mathbf{x}, \mathbf{x}^*)$  for a given  $\mathbf{x}^* \in \mathbb{R}^d$ .

The Hilbert space  $\mathcal{H}$  is defined as a Reproducing Kernel Hilbert Space (RKHS) Berlinet and Thomas-Agnan (2004) with reproducing Kernel  $\mathbf{k}$  because it verifies, for any  $f \in \mathcal{H}$  and  $\mathbf{x} \in \mathbb{R}^d$ ,

$$\langle f, \mathbf{k}(\mathbf{x}, \cdot) \rangle_{\mathcal{H}} = f(\mathbf{x}), \quad (19)$$

where  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  is the dot product associated to the Hilbert space  $\mathcal{H}$ .

It is shown by the Representer theorem Schölkopf and Smola (2001) that any minimizer to the empirical risk of the function  $f \in \mathcal{H}$  admits a representation of the form

$$\hat{f}(\mathbf{x}) = \sum_{i=1}^n \alpha_i \mathbf{k}(\mathbf{x}^{(i)}, \mathbf{x}), \quad (20)$$

where  $\boldsymbol{\alpha} \in \mathbb{R}^n$ .

In the following, we consider the Matérn anisotropic geometric kernel

$$\mathbf{k}_{\sigma^2, \boldsymbol{\theta}}(\mathbf{x}, \mathbf{x}') = r_{\sigma^2, \boldsymbol{\theta}}^\nu \left( \sqrt{\sum_{j=1}^d \frac{|x_j - x'_j|^2}{\theta_j^2}} \right), \quad (21)$$

where  $r$  is a Matérn kernel in  $\mathbb{R}$ ,  $\nu$  is usually taken for  $\nu = \frac{5}{2}$  (Matérn 5/2) or  $\nu \rightarrow \infty$ ,  $\sigma^2$  is the variance amplitude,  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_d) \in \mathbb{R}^d$  the length-scale vector, and we assume that each  $\tau_t$  belongs to  $\mathcal{H}$ , the Reproducing Kernel Hilbert Space (RKHS) with reproducing kernel  $\mathbf{k}_{\sigma^2, \boldsymbol{\theta}}$  in such way that

$$\mathcal{F} = \left\{ \left\{ \tau(\mathbf{x}) = \sum_{i=1}^n \alpha_{t,i} \mathbf{k}_{\sigma^2, \boldsymbol{\theta}}(\mathbf{x}^{(i)}, \mathbf{x}) \right\}_{t \neq t_1} / \boldsymbol{\alpha}_t = (\alpha_{t,1}, \dots, \alpha_{t,n})^\top \in \mathbb{R}^n \right\}. \quad (22)$$

Similarly to linear regression models, it is possible to use a matrix notation  $\tau_{t_k}(\mathbf{X}) = \mathbf{K} \boldsymbol{\alpha}_k$  where  $\mathbf{K}_{ij} = (\mathbf{k}_{\sigma^2, \boldsymbol{\theta}}(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}))$  is the Gram matrix of  $\mathbf{k}_{\sigma^2, \boldsymbol{\theta}}$ .

For a fixed hyperparameters  $(\sigma^2, \boldsymbol{\theta})$ , we prove immediately that the R-learning problem in (5) is similar to a linear regression problem. Therefore, by Proposition 4.6, the coefficients  $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_K)$  satisfy

$$\hat{\boldsymbol{\alpha}} = \hat{\boldsymbol{\alpha}}_{\sigma^2, \boldsymbol{\theta}} = \mathbf{A}_{\sigma^2, \boldsymbol{\theta}}^+ \mathbf{a}_{\sigma^2, \boldsymbol{\theta}}, \quad (23)$$

where  $\mathbf{A}_{\sigma^2, \boldsymbol{\theta}}^+$  is the Moore–Penrose inverse of  $\mathbf{A}_{\sigma^2, \boldsymbol{\theta}}$  and  $\mathbf{a}_{\sigma^2, \boldsymbol{\theta}} = (\mathbf{a}_2^\top, \dots, \mathbf{a}_K^\top)^\top$  such that

$$\begin{aligned} (\mathbf{a}_{\sigma^2, \boldsymbol{\theta}})_j &= \frac{1}{n} \mathbf{K}_{\sigma_j}^\top \mathbf{D}_{\bar{T}_j} \bar{Y} \in \mathbb{R}^n, \\ (\mathbf{B}_{\sigma^2, \boldsymbol{\theta}})_j &= \frac{1}{n} \mathbf{K}_{\sigma_j}^\top \mathbf{D}_{\bar{T}_j}^2 \mathbf{K}_{\sigma_j} \in \mathbb{R}^{n \times n}, \\ (\mathbf{C}_{\sigma^2, \boldsymbol{\theta}})_{ij} &= \frac{1}{n} \mathbf{K}_{\sigma_i}^\top \mathbf{D}_{\bar{T}_i} \mathbf{D}_{\bar{T}_j} \mathbf{K}_{\sigma_j} \in \mathbb{R}^{n \times n}. \end{aligned}$$

Finally, by considering  $\hat{\tau}_k(\mathbf{x}) = \sum_{i=1}^n (\hat{\boldsymbol{\alpha}}_k)_i \mathbf{k}_{\sigma^2, \boldsymbol{\theta}}(\mathbf{x}^{(i)}, \mathbf{x})$ , one can obtain the optimal hyperparameters  $(\sigma^2, \boldsymbol{\theta})$  by solving the problem :

$$(\hat{\sigma}^2, \hat{\boldsymbol{\theta}}) = \arg \min_{(\sigma^2, \boldsymbol{\theta})} \left\{ \frac{1}{n} \sum_{i=1}^n \left[ (Y_{\text{obs},i} - \hat{m}(\mathbf{X}_i)) - \sum_{k=2}^K (\mathbf{1}\{T_i = t_k\} - \hat{r}(t_k, \mathbf{X}_i)) \hat{\tau}_k(\mathbf{X}_i) \right]^2 \right\}. \quad (24)$$

This problem admits a explicit solution for  $\hat{\sigma}^2$  by direct calculations as in the proof of Proposition 4.6 such that

$$\hat{\sigma}^2 = \frac{\sum_{k=2}^K (\bar{Y}^\top \mathbf{D}_{\bar{T}_k} \mathbf{R}_{\boldsymbol{\theta}}) \boldsymbol{\alpha}_k}{\sum_{k,k'=2}^K \boldsymbol{\alpha}_k^\top (\mathbf{R}_{\boldsymbol{\theta}}^\top \mathbf{D}_{\bar{T}_k} \mathbf{D}_{\bar{T}_{k'}} \mathbf{R}_{\boldsymbol{\theta}}) \boldsymbol{\alpha}_{k'}}, \quad (25)$$

whereas the optimal length-scale vector  $\hat{\boldsymbol{\theta}}$  can be obtained numerically by running, for example, a multistart gradient descent algorithm or multistart BFGS method.

$$\hat{\boldsymbol{\theta}} = \arg \min_{\boldsymbol{\theta}} \left\{ \frac{1}{n} \sum_{i=1}^n \left[ (Y_{\text{obs},i} - \hat{m}(\mathbf{X}_i)) - \sum_{k=2}^K (\mathbf{1}\{T_i = t_k\} - \hat{r}(t_k, \mathbf{X}_i)) \hat{\tau}_{k, \hat{\sigma}^2}(\mathbf{X}_i) \right]^2 \right\}. \quad (26)$$

## B Error estimation of two-step meta-learners.

In the following subsection, we will analyze the error estimation of each two-step meta-learner. Let us assume that our observations are generated from the a function  $f$  respecting the two causal assumptions (3.1-3.2) such that for all  $t \in \mathcal{T}$

$$Y(t) = f(t, \mathbf{X}) + \epsilon \text{ with } \epsilon \sim \mathcal{N}(0, \sigma^2) \quad (27)$$

Each unit  $i$  has the following observed and potential outcomes

$$\begin{aligned} Y_{\text{obs},i} &= Y_i(T_i) = f(T_i, \mathbf{X}_i) + \epsilon_i, \\ Y_i(t) &= f(t, \mathbf{X}_i) + \epsilon_i(t), \\ Y_i(t_1) &= f(t_1, \mathbf{X}_i) + \epsilon_i(t_1). \end{aligned} \quad (28)$$

where  $\epsilon_i(t_1)$  and  $\epsilon_i(t_1)$  are some Gaussian noise like  $\epsilon$ .

*Remark.* We recall that  $(Y_i(t))_{1 \leq i \leq n}$  and  $(Y_i(t_1))_{1 \leq i \leq n}$  are virtual vectors and cannot be observed.

The CATE model  $\tau_t$  for each  $t \in \mathcal{T}$  can be written as

$$\begin{aligned}\tau_t(\mathbf{x}) &= \mathbb{E}(Y(t) - Y(t_1) \mid \mathbf{X} = \mathbf{x}) \\ &= \mathbb{E}(f(t, \mathbf{X}) - f(t_1, \mathbf{X}) + \epsilon^* \mid \mathbf{X} = \mathbf{x}) \\ &= f(t, \mathbf{x}) - f(t_1, \mathbf{x})\end{aligned}\tag{29}$$

with  $\epsilon^*$  is a noise independent from  $\mathbf{X}$  and satisfying  $\mathbb{E}(\epsilon^*) = 0$ .

We want to estimate the CATE for each  $t \neq t_1$  using linear regression model. We will assume the existence a vector  $\beta_t^* \in \mathbb{R}^p$  such that  $f(t, \mathbf{x}) = \sum_{j=0}^{p-1} \beta_{t,j}^* f_j(\mathbf{x})$  (e.g. polynomial functions  $f_j(\mathbf{x}) = (\mathbf{x}_k^j)_{1 \leq k \leq d}$ ).

Under the previous assumption, we write  $\tau_t(\mathbf{X}) = f(t, \mathbf{X}) - f(t_1, \mathbf{X}) = \mathbf{H}\beta^*$  where  $\beta^* = \beta_t^* - \beta_{t_1}^*$  and  $\mathbf{H} = (\mathbf{H}_{ij}) \in \mathbb{R}^{n \times p}$  is the regression matrix, assumed to be full rank matrix, such that  $\mathbf{H}_{ij} = f_j(\mathbf{X}_i)$  for  $i = 1, \dots, n$  and  $j = 0, \dots, p-1$ .

With pseudo-outcome meta-learners, we consider a random Variable  $Z_t$  for a fixed  $t$  such that

$$Z_{t,i} = A_t(T_i, \mathbf{X}_i)Y_{\text{obs},i} + B_t(T_i, \mathbf{X}_i), \quad i = 1, \dots, n,$$

where the functions  $A_t(T, \mathbf{X})$  and  $B_t(T, \mathbf{X})$  are given for each pseudo-outcome meta-learners.

The regression coefficient  $\hat{\beta}_t$  are given by the Ordinary Least Squares (OLS) method

$$\hat{\beta}_t = (\mathbf{H}^\top \mathbf{H})^{-1} \mathbf{H}^\top \mathbf{z}_t,\tag{30}$$

where  $\mathbf{z}_t = (Z_{t,i})_{1 \leq i \leq n}$ . Thus,

$$\begin{aligned}\hat{\beta}_t &= (\mathbf{H}^\top \mathbf{H})^{-1} \mathbf{H}^\top \mathbf{z}_t \\ &= (\mathbf{H}^\top \mathbf{H})^{-1} \mathbf{H}^\top (A_t(T_i, \mathbf{X}_i)Y_{\text{obs},i} + B_t(T_i, \mathbf{X}_i))_{i=1}^n \\ &= (\mathbf{H}^\top \mathbf{H})^{-1} \mathbf{H}^\top (A_t(T_i, \mathbf{X}_i)f(T_i, \mathbf{X}_i) + B_t(T_i, \mathbf{X}_i) + A_t(T_i, \mathbf{X}_i)\epsilon_i)_{i=1}^n \\ &= (\mathbf{H}^\top \mathbf{H})^{-1} \mathbf{H}^\top (\tau_t(\mathbf{x}) + A_t(T_i, \mathbf{X}_i)f(T_i, \mathbf{X}_i) - \tau_t(\mathbf{x}) + B_t(T_i, \mathbf{X}_i) + A_t(T_i, \mathbf{X}_i)\epsilon_i)_{i=1}^n \\ &= (\mathbf{H}^\top \mathbf{H})^{-1} \mathbf{H}^\top (\mathbf{H}\beta_t^* + A_t(T_i, \mathbf{X}_i)f(T_i, \mathbf{X}_i) - f(t, \mathbf{X}_i) + B_t(T_i, \mathbf{X}_i)A_t(T_i, \mathbf{X}_i)\epsilon_i)_{i=1}^n \\ &= \beta_t^* + (\mathbf{H}^\top \mathbf{H})^{-1} \mathbf{H}^\top (A_t(T_i, \mathbf{X}_i)f(T_i, \mathbf{X}_i) - \tau_t(\mathbf{x}) + B_t(T_i, \mathbf{X}_i) + A_t(T_i, \mathbf{X}_i)\epsilon_i)_{i=1}^n \\ &= \beta_t^* + (\mathbf{H}^\top \mathbf{H})^{-1} \mathbf{H}^\top \tilde{\epsilon}_t\end{aligned}$$

where  $\tilde{\epsilon}_i = \psi_t(T_i, \mathbf{X}_i) + A_t(T_i, \mathbf{X}_i)\epsilon_i$  and  $\psi_t(T_i, \mathbf{X}_i) = A_t(T_i, \mathbf{X}_i)f(T_i, \mathbf{X}_i) - \tau_t(\mathbf{x}) + B_t(T_i, \mathbf{X}_i)$  to simplify notations.

Let us consider the random vector  $\mathbf{Z}^{(n)}$  such that

$$\mathbf{Z}_t^{(n)} = \left( \frac{1}{n}(\mathbf{H}^\top \tilde{\epsilon}_t)_1, \dots, \frac{1}{n}(\mathbf{H}^\top \tilde{\epsilon}_t)_p, \frac{1}{n}(\mathbf{H}^\top \mathbf{H})_{11}, \dots, \frac{1}{n}(\mathbf{H}^\top \mathbf{H})_{pp} \right)^\top \in \mathbb{R}^{p+p^2},\tag{31}$$

that allows us to write  $\hat{\beta}_t$  as

$$\begin{aligned}\hat{\beta}_t &= \beta_t^* + (\mathbf{H}^\top \mathbf{H})^{-1} \mathbf{H}^\top \tilde{\epsilon}_t \\ &= \beta_t^* + \left( \frac{1}{n} \mathbf{H}^\top \mathbf{H} \right)^{-1} \left( \frac{1}{n} \mathbf{H}^\top \tilde{\epsilon}_t \right) \\ &= \beta_t^* + \phi(\mathbf{Z}_t^{(n)})\end{aligned}\tag{32}$$

where  $\phi : \mathbb{R}^{p+p^2} \rightarrow \mathbb{R}^p$  is a  $\mathcal{C}^1$ -function.

In order to apply the Central Limit Theorem (CLT) later, we show that the vector  $\mathbf{Z}_t^{(n)}$  can be written as sum of *i.i.d* random vectors  $\mathbf{Z}_{t,i}$ .

$$\begin{aligned}\mathbf{Z}_t^{(n)} &= \left( \frac{1}{n}(\mathbf{H}^\top \tilde{\epsilon})_1, \dots, \frac{1}{n}(\mathbf{H}^\top \tilde{\epsilon})_p, \frac{1}{n}(\mathbf{H}^\top \mathbf{H})_{11}, \dots, \frac{1}{n}(\mathbf{H}^\top \mathbf{H})_{pp} \right)^\top \in \mathbb{R}^{p+p^2} \\ &= \left( \frac{1}{n} \sum_{i=1}^n \mathbf{H}_{i1} \tilde{\epsilon}_i, \dots, \mathbf{H}_{ip} \tilde{\epsilon}_i, \frac{1}{n} \sum_{i=1}^n \mathbf{H}_{i1} \mathbf{H}_{i1}, \dots, \frac{1}{n} \sum_{i=1}^n \mathbf{H}_{ip} \mathbf{H}_{ip} \right)^\top \\ &= \frac{1}{n} \sum_{i=1}^n (\mathbf{H}_{i1} \tilde{\epsilon}_i, \dots, \mathbf{H}_{ip} \tilde{\epsilon}_i, \mathbf{H}_{i1} \mathbf{H}_{i1}, \dots, \mathbf{H}_{ip} \mathbf{H}_{ip})^\top = \frac{1}{n} \sum_{i=1}^n \mathbf{Z}_{t,i}\end{aligned}\tag{33}$$

The mean  $\mathbf{m}$  of the vector  $\mathbf{Z}_t^{(n)}$  satisfies

$$\begin{aligned}\mathbf{m} &= \mathbb{E}(\mathbf{Z}_t^{(n)}) = \frac{1}{n} \sum_{i=1}^n \mathbb{E}(\mathbf{Z}_{t,i}) = \mathbb{E}(\mathbf{Z}_{t,i}) \\ &= \left( h_1, \dots, h_p, F_{11}, \dots, F_{pp} \right)^\top\end{aligned}\quad (34)$$

where

$$\begin{aligned}h_j &= \mathbb{E}[f_j(\mathbf{X})(\psi_t(T, \mathbf{X}) + A_t(T, \mathbf{X})\epsilon)] = \mathbb{E}(f_j(\mathbf{X})\psi_t(T, \mathbf{X})) \\ F_{jj'} &= \mathbb{E}(f_j(\mathbf{X})f_{j'}(\mathbf{X})),\end{aligned}\quad (35)$$

and a covariance matrix  $\mathbf{C}$  with entries

$$\begin{aligned}\mathbf{C}_{jj'} &= \text{Cov}(\mathbf{Z}_{t,j}, \mathbf{Z}_{t,j'}) = \mathbb{E}(Z_j, Z_{j'}) - \mathbb{E}(Z_{t,j})\mathbb{E}(Z_{t,j'}) \\ &= \begin{cases} \mathbb{E}(f_j(\mathbf{X})f_{j'}(\mathbf{X})(\psi_t(T, \mathbf{X}) + A_t(T, \mathbf{X})\epsilon)^2) - h_j h_{j'} & \text{if } j, j' \in \{1, \dots, p\} \\ \mathbb{E}(f_k(\mathbf{X})f_{k'}(\mathbf{X})f_l(\mathbf{X})f_{l'}(\mathbf{X})) - F_{kk'}F_{ll'} & \text{if } j, j' \in \{p+1, \dots, p^2\} \\ \mathbb{E}(f_k(\mathbf{X})f_{k'}(\mathbf{X})f_j(\mathbf{X})(\psi_t(T, \mathbf{X}) + A_t(T, \mathbf{X})\epsilon)) - h_j F_{kk'} & \text{otherwise.} \end{cases} \\ &= \begin{cases} \mathbb{E}(f_j(\mathbf{X})f_{j'}(\mathbf{X})\psi_t^2(T, \mathbf{X})) + \sigma^2\mathbb{E}(f_j(\mathbf{X})f_{j'}(\mathbf{X})A_t^2(T, \mathbf{X})) - h_j h_{j'} & j, j' \in \{1, \dots, p\} \\ \mathbb{E}(f_k(\mathbf{X})f_{k'}(\mathbf{X})f_l(\mathbf{X})f_{l'}(\mathbf{X})) - F_{kk'}F_{ll'} & \text{if } j, j' \in \{p+1, \dots, p^2\} \\ \mathbb{E}(f_k(\mathbf{X})f_{k'}(\mathbf{X})f_j(\mathbf{X})\psi_t(T, \mathbf{X})) - h_j F_{kk'} & \text{otherwise.} \end{cases}\end{aligned}\quad (36)$$

where  $k, k' = \eta^{-1}(j)$  such that  $\eta$  is the correspondence indexes map between  $\mathbf{m}$  and  $F$  in  $\mathbf{m}_j = F_{kk'}$  when  $j \geq p+1$ .

By considering now the vector

$$\mathbf{S}^{(n)} = \sqrt{n}(\mathbf{Z}_t^{(n)} - \mathbf{m}) = \frac{1}{\sqrt{n}} \sum_{i=1}^n (\mathbf{Z}_{t,i} - \mathbf{m}),\quad (37)$$

one can show by the multivariate Central Limit Theorem (CLT) that

$$\mathbf{S}^{(n)} = \sqrt{n}(\mathbf{Z}^{(n)} - \mathbf{m}) \xrightarrow{\mathcal{L}} \mathcal{N}(\mathbf{0}, \mathbf{C}).\quad (38)$$

Which allows us to write  $\widehat{\beta}_t$  as function of  $\mathbf{S}^{(n)}$  and  $\mathbf{m}$ . Indeed,

$$\begin{aligned}\widehat{\beta}_t &= \beta_t^* + (\mathbf{H}^\top \mathbf{H})^{-1} \mathbf{H}^\top \tilde{\epsilon} \\ &= \beta_t^* + \phi(\mathbf{Z}^{(n)}) \\ &= \beta_t^* + \phi(\mathbf{m} + \mathbf{S}^{(n)}/\sqrt{n}) \\ &= \beta_t^* + \Phi(\mathbf{S}^{(n)}, \mathbf{m}),\end{aligned}\quad (39)$$

where  $\Phi : \mathbb{R}^{p+p^2} \times \mathbb{R}^{p+p^2} \rightarrow \mathbb{R}^p$  is also  $\mathcal{C}^1$ -function.

Since  $\sqrt{n}(\mathbf{S}^{(n)} - \mathbf{0}) \xrightarrow{\mathcal{L}} \mathcal{N}(\mathbf{0}, \mathbf{C})$ , one obtains by the Delta method

$$\sqrt{n}[\Phi(\mathbf{S}^{(n)}, \mathbf{m}) - \Phi(\mathbf{0}, \mathbf{m})] \xrightarrow{\mathcal{L}} \mathcal{N}\left(\mathbf{0}, J_\Phi^{(1)}(\mathbf{0}, \mathbf{m})^\top \mathbf{C} J_\Phi^{(1)}(\mathbf{0}, \mathbf{m})\right),\quad (40)$$

where  $J_\Phi^{(1)}(\mathbf{0}, \mathbf{m})$  is the Jacobian matrix at the first  $p+p^2$  coordinates of  $\Phi$  at  $(\mathbf{0}, \mathbf{m})$ .

By denoting  $\mathbf{g}_n$ , a Gaussian noise with zero-mean and covariance matrix of  $\mathbf{C}' = J_\Phi^{(1)}(\mathbf{0}, \mathbf{m})^\top \mathbf{C} J_\Phi^{(1)}(\mathbf{0}, \mathbf{m})$ , the previous equation is equivalent to

$$\widehat{\beta}_t = \beta_t^* + \Phi(\mathbf{S}_n, \mathbf{m}) \approx \beta_t^* + \Phi(\mathbf{0}, \mathbf{m}) + \mathbf{g}_n/\sqrt{n}.\quad (41)$$

Therefore, for  $n$  big enough :

$$\mathbb{E}(\widehat{\beta}_t) \approx \beta_t^* + \Phi(\mathbf{0}, \mathbf{m}).\quad (42)$$

and,

$$\mathbb{V}(\widehat{\beta}_t) \approx \frac{1}{n} J_{\Phi}^{(1)}(\mathbf{0}, \mathbf{m})^\top \mathbf{C} J_{\Phi}^{(1)}(\mathbf{0}, \mathbf{m}). \quad (43)$$

This results holds whether the nuisance parameters in  $A_t$  and  $B_t$  are well-specified or not, so there is no guarantee that  $\Phi(\mathbf{0}, \mathbf{m}) = 0$  and the estimator  $\widehat{\beta}_t$  may be biased.

In the following, we assume that the nuisance parameters in  $A_t$  and  $B_t$  are well-specified i.e.  $\mathbb{E}(\psi_t(T, \mathbf{X})) | \mathbf{X} = \mathbf{x}) = 0$  in such way that  $\mathbb{E}(Z_t | \mathbf{X} = \mathbf{x}) = \mu_t(\mathbf{x})$ , or equivalently,  $\mathbb{E}(\mathbf{H}^\top \tilde{\epsilon}) = \mathbf{0}$ . Consequently, the estimator of  $\widehat{\beta}_t$  is unbiased. In this case, computing the variance  $\mathbb{V}(\widehat{\beta}_t)$  becomes much easier and explicit.

On the one hand, by the multivariate Central Theorem Limit (CTL)

$$\frac{1}{\sqrt{n}} \mathbf{H}^\top \tilde{\epsilon} \xrightarrow{\mathcal{L}} \mathcal{N}(\mathbf{0}, \Sigma) \quad (44)$$

which is equivalent to

$$\frac{1}{\sqrt{n}} \mathbf{H}^\top \tilde{\epsilon} \approx \mathbf{g}_n, \quad (45)$$

where  $\mathbf{g}_n$  is a Gaussian noise with zero-mean and covariance matrix of  $\Sigma$  with entries

$$\begin{aligned} \Sigma_{jj'} &= \mathbb{E}[f_j(\mathbf{X})f_{j'}(\mathbf{X})(\psi(T, \mathbf{X}) + A_t(T, \mathbf{X})\epsilon)^2] \\ &= \mathbb{E}(f_j(\mathbf{X})f_{j'}(\mathbf{X})\psi_t^2(T, \mathbf{X})) + \sigma^2 \mathbb{E}(f_j(\mathbf{X})f_{j'}(\mathbf{X})A_t^2(T, \mathbf{X})). \end{aligned} \quad (46)$$

On the other hand, by the law of large numbers, we have  $1/n(\mathbf{H}^\top \mathbf{H}) \xrightarrow{a.s.} \mathbf{F}$ , thus  $1/n(\mathbf{H}^\top \mathbf{H}) \xrightarrow{P} \mathbf{F}$ . Since  $\mathbf{F}$  is invertible, then

$$n(\mathbf{H}^\top \mathbf{H})^{-1} \xrightarrow{P} \mathbf{F}^{-1}, \quad (47)$$

where  $\mathbf{F} = (F_{jj'})_{1 \leq j, j' \leq p}$  and  $F_{jj'} = \mathbb{E}(f_j(\mathbf{X})f_{j'}(\mathbf{X}))$ .

By Slutsky's theorem,

$$\begin{aligned} \sqrt{n}(\widehat{\beta}_t - \beta_t^*) &= n(\mathbf{H}^\top \mathbf{H})^{-1} \cdot 1/\sqrt{n} \mathbf{H}^\top \tilde{\epsilon} \\ &\xrightarrow{\mathcal{L}} \mathcal{N}(\mathbf{0}, \mathbf{F}^{-1} \Sigma \mathbf{F}^{-1}) \end{aligned} \quad (48)$$

which leads to

$$\begin{aligned} \mathbb{E}(\widehat{\beta}_t) &\approx \beta_t^*, \\ \mathbb{V}(\widehat{\beta}_t) &\approx \frac{1}{n} \mathbf{F}^{-1} \Sigma \mathbf{F}^{-1}. \end{aligned} \quad (49)$$

The determinant of the variance matrix, also known as the generalized variance by Wilks (1932, 1967) is usually used as a scalar measure of overall multidimensional scatter and can be useful to compare the variance of each meta-learner.

In our case, comparing the generalized variance is equivalent to comparing  $\det\left(\frac{1}{n}\Sigma\right)$  of each pseudo-outcome meta-learner since

$$\det(\mathbb{V}(\widehat{\beta}_t)) = (\det \mathbf{F}^{-1})^2 \det\left(\frac{1}{n}\Sigma\right) = \frac{1}{(\det \mathbf{F})^2} \det\left(\frac{1}{n}\Sigma\right), \quad (50)$$

with, obviously,  $\det(\Sigma) > 0$  because  $\Sigma$  is symmetric positive definite.

In addition to the assumption (3.2) which states that,  $0 < r_{\min} \leq r(t, \mathbf{X})$ , we will assume that the potential outcomes function  $f$  and models  $\mu_w$  are bounded, i.e. there exists  $C > 0$  such that  $|\mu_w(\mathbf{x})|, |f(w, \mathbf{x})| \leq C$  for all  $w \in \mathcal{T}$  and  $\mathbf{x} \in \mathbb{R}^d$ .

### B.1 Error estimation of the M-learner

**Lemma B.1.** *If  $X_1, \dots, X_m$  is a sequence of random variables and  $b > 1$ , then*

$$\begin{aligned} \left| \mathbb{E} \left[ \left( \sum_{i=1}^m X_i \right)^2 \right] \right| &\leq m \sum_{i=1}^m \mathbb{E} \left[ |X_i^2| \right], \\ \left| \mathbb{E} \left[ \left( \sum_{i=1}^m X_i \right)^b \right] \right| &\leq m^{(b-1)} \sum_{i=1}^m \mathbb{E} \left[ |X_i^2| \right]. \end{aligned} \quad (51)$$

*Proof.* By induction and Minkowski inequality. □

Let  $a, b > 1$  such that  $1/a + 1/b = 1$  and we assume that  $f_j(\mathbf{X}) \in L^a$  (i.e.  $f_j(\mathbf{X})$  has all possible finite moments) for all  $j \in \{1, \dots, p\}$ . By Hölder inequality we show that for the M-learner :

$$\begin{aligned} |\mathbb{E}(f_j(\mathbf{X})f_{j'}(\mathbf{X})\psi_t^2(T, \mathbf{X}))| &\leq |\mathbb{E}(f_j^a(\mathbf{X})f_{j'}^a(\mathbf{X}))|^{1/a} \cdot |\mathbb{E}(\psi_t^{2b}(T, \mathbf{X}))|^{1/b} \quad (\text{Hölder}) \\ &\leq \delta_{jj'}^{(a)} \left( 2^{2b-1} \mathbb{E} \left[ \left( \frac{\mathbf{1}\{T=t\}}{r(t, \mathbf{X})} - 1 \right)^{2b} f^{2b}(t, \mathbf{X}) \right. \right. \\ &\quad \left. \left. + \left( \frac{\mathbf{1}\{T=t_1\}}{r(t_1, \mathbf{X})} - 1 \right)^{2b} f^{2b}(t_1, \mathbf{X}) \right] \right)^{1/b} \quad (\text{Lemma B.1, with } \delta_{jj'}^{(a)} = |\mathbb{E}(f_j^a(\mathbf{X})f_{j'}^a(\mathbf{X}))|^{1/a}) \\ &\leq 2^{2(2b-1)/b} \delta_{jj'}^{(a)} \left( \mathbb{E} \left[ 2^{2b-1} \left( \frac{\mathbf{1}\{T=t\}}{r^{2b}(t, \mathbf{X})} + 1 \right) f^{2b}(t, \mathbf{X}) \right] \right. \\ &\quad \left. + \mathbb{E} \left[ 2^{2b-1} \left( \frac{\mathbf{1}\{T=t_1\}}{r^{2b}(t_1, \mathbf{X})} + 1 \right) f^{2b}(t_1, \mathbf{X}) \right] \right)^{1/b} \quad (\text{Lemma B.1}) \\ &\leq 2^{2(2b-1)/b} \delta_{jj'}^{(a)} \left( \mathbb{E} \left[ \mathbb{E} \left( \frac{\mathbf{1}\{T=t\}}{r^{2b}(t, \mathbf{X})} + 1 \right) \mid \mathbf{X} \right) f^{2b}(t, \mathbf{X}) \right] \right. \\ &\quad \left. + \mathbb{E} \left[ \mathbb{E} \left( \frac{\mathbf{1}\{T=t_1\}}{r^{2b}(t_1, \mathbf{X})} + 1 \right) \mid \mathbf{X} \right) f^{2b}(t_1, \mathbf{X}) \right] \right)^{1/b} \\ &\leq 2^{2(2b-1)/b} \delta_{jj'}^{(a)} \left( \mathbb{E} \left[ \left( \frac{1}{r^{2b-1}(t, \mathbf{X})} + 1 \right) f^{2b}(t, \mathbf{X}) \right] \right. \\ &\quad \left. + \mathbb{E} \left[ \left( \frac{1}{r^{2b-1}(t_1, \mathbf{X})} + 1 \right) f^{2b}(t_1, \mathbf{X}) \right] \right)^{1/b} \\ &\leq 2^{2(2b-1)/b} \delta_{jj'}^{(a)} \left( \frac{1}{r_{\min}^{2b-1}} + 1 \right)^{1/b} \left( C^{2b} + C^{2b} \right)^{1/b} \quad (\text{Bounding } r \text{ and } f) \\ &\leq 2^{2(2b-1)/b} \delta_{jj'}^{(a)} \left( \frac{1}{r_{\min}^{2b-1}} + \frac{1}{r_{\min}^{2b-1}} \right)^{1/b} 2^{1/b} C^b \\ &\leq 2^{2(2b-1)/b} \delta_{jj'}^{(a)} \frac{2^{1/b}}{r_{\min}^{(2b-1)/b}} 2^{1/b} C^b \\ &\leq 2^4 \delta_{jj'}^{(a)} \frac{1}{r_{\min}^{(2b-1)/b}} C^b = \frac{16}{r_{\min}^{(2b-1)/b}} \delta_{jj'}^{(a)} C^b \end{aligned} \quad (52)$$

On the other term, one obtains similarly:

$$\begin{aligned}
 |\mathbb{E}(f_j(\mathbf{X})f_{j'}(\mathbf{X})A_t^2(T, \mathbf{X}))| &\leq |\mathbb{E}(f_j^a(\mathbf{X})f_{j'}^a(\mathbf{X}))|^{1/a} \cdot |\mathbb{E}(A_t^{2b}(T, \mathbf{X}))|^{1/b} \quad (\text{Hölder}) \\
 &\leq \delta_{jj'}^{(a)} |\mathbb{E}(A_t^{2b}(T, \mathbf{X}))|^{1/b} \\
 &\leq \delta_{jj'}^{(a)} \left( 2^{2b-1} \mathbb{E}\left(\frac{\mathbf{1}\{T=t\}}{r(t, \mathbf{X})}\right)^{2b} + \mathbb{E}\left(\frac{\mathbf{1}\{T=t_1\}}{r(t_1, \mathbf{X})}\right)^{2b} \right)^{1/b} \quad (\text{Lemma B.1}) \\
 &\leq 2^{(2b-1)/b} \sigma^2 \delta_{jj'}^{(a)} \left( \mathbb{E}\left(\frac{\mathbf{1}\{T=t\}}{r^{2b}(t, \mathbf{X})}\right) + \mathbb{E}\left(\frac{\mathbf{1}\{T=t_1\}}{r^{2b}(t_1, \mathbf{X})}\right) \right)^{1/b} \\
 &\leq 2^{(2b-1)/b} \sigma^2 \delta_{jj'}^{(a)} \left(\frac{2}{r_{\min}^{2b-1}}\right)^{1/b} = \frac{4}{r_{\min}^{(2b-1)/b}} \sigma^2 \delta_{jj'}^{(a)}
 \end{aligned} \tag{53}$$

Thus, by combining the two terms, one gets:

$$\begin{aligned}
 \left| \Sigma_{jj'}^{(M)} \right| &\leq |\mathbb{E}(f_j(\mathbf{X})f_{j'}(\mathbf{X})\psi_t^2(T, \mathbf{X}))| + \sigma^2 |\mathbb{E}(f_j(\mathbf{X})f_{j'}(\mathbf{X})A_t^2(T, \mathbf{X}))| \\
 &\leq \frac{16}{r_{\min}^{(2b-1)/b}} \delta_{jj'}^{(a)} C^b + \frac{4}{r_{\min}^{(2b-1)/b}} \sigma^2 \delta_{jj'}^{(a)} \\
 &\leq \frac{1}{r_{\min}^{(2b-1)/b}} (16 C^b + 4\sigma^2) \delta_*^{(b)}
 \end{aligned} \tag{54}$$

where  $\delta_*^{(b)} = \max_{j,j'} \left| \mathbb{E}(f_j^{b/(b-1)}(\mathbf{X})f_{j'}^{b/(b-1)}(\mathbf{X})) \right|^{(b-1)/b} = \max_{j,j'} \delta_{jj'}^{(a)}$ .

Therefore, for all  $\epsilon = b - 1 > 0$ , there exists  $C_M = 4 C + \sigma^2$  such that

$$\left| \Sigma_{jj'}^{(M)} \right| \leq 4r_{\min}^{1/(1+\epsilon)-2} \delta_*^{(1+\epsilon)} C_M. \tag{55}$$

In particular, if  $\epsilon \ll 1$  then  $1/(1+\epsilon) - 2 \approx -(1+\epsilon)$  and

$$\left| \Sigma_{jj'}^{(M)} \right| \leq \frac{4}{r_{\min}^{1+\epsilon}} \delta_*^{(1+\epsilon)} C_M. \tag{56}$$

## B.2 Error estimation of the DR-learner.

In this case, we have  $A_t(T, \mathbf{X}) = \frac{\mathbf{1}\{T=t\}}{r(t, \mathbf{X})} - \frac{\mathbf{1}\{T=t_1\}}{r(t, \mathbf{X})}$  and  $B_t(T, \mathbf{X}) = \mu_t(\mathbf{X}) - \mu_{t_1}(\mathbf{X}) - \left( \frac{\mathbf{1}\{T=t\}}{r(t, \mathbf{X})} - \frac{\mathbf{1}\{T=t_1\}}{r(t, \mathbf{X})} \right) \mu_T(\mathbf{X})$ . We need just to compute the upper bound of  $\mathbb{E}(f_j(\mathbf{X})f_{j'}(\mathbf{X})\psi_t^2(T, \mathbf{X}))$  such that

$$\begin{aligned}
 \psi_t(T, \mathbf{X}) &= A_t(T, \mathbf{X})f(T, \mathbf{X}) - \tau_t(\mathbf{x}) + B_t(T, \mathbf{X}) \\
 &= \left( \frac{\mathbf{1}\{T=t\}}{r(t, \mathbf{X})} - 1 \right) f(t, \mathbf{X}) - \left( \frac{\mathbf{1}\{T=t_1\}}{r(t_1, \mathbf{X})} - 1 \right) f(t_1, \mathbf{X}) + \mu_t(\mathbf{X}) \left( 1 - \frac{\mathbf{1}\{T=t\}}{r(t, \mathbf{X})} \right) \\
 &\quad - \mu_{t_1}(\mathbf{X}) \left( 1 - \frac{\mathbf{1}\{T=t_1\}}{r(t_1, \mathbf{X})} \right) \\
 &= \left( \frac{\mathbf{1}\{T=t\}}{r(t, \mathbf{X})} - 1 \right) (f(t, \mathbf{X}) - \mu_t(\mathbf{X})) - \left( \frac{\mathbf{1}\{T=t_1\}}{r(t_1, \mathbf{X})} - 1 \right) (f(t_1, \mathbf{X}) - \mu_{t_1}(\mathbf{X}))
 \end{aligned} \tag{57}$$

Similarly to the previous calculus, we show that for the DR-learner

$$\begin{aligned}
 |\mathbb{E}(f_j(\mathbf{X})f_{j'}(\mathbf{X})\psi_t^2(T, \mathbf{X}))| &\leq |\mathbb{E}(f_j^a(\mathbf{X})f_{j'}^a(\mathbf{X}))|^{1/a} \cdot |\mathbb{E}(\psi_t^{2b}(T, \mathbf{X}))|^{1/b} \quad (\text{Hölder}) \\
 &\leq \delta_{jj'}^{(a)} \left( 2^{2b-1} \mathbb{E} \left[ \left( \frac{\mathbf{1}\{T=t\}}{r(t, \mathbf{X})} - 1 \right)^{2b} (f(t, \mathbf{X}) - \mu_t(\mathbf{X}))^{2b} \right. \right. \\
 &\quad \left. \left. + \left( \frac{\mathbf{1}\{T=t_1\}}{r(t, \mathbf{X})} - 1 \right)^{2b} (f(t_1, \mathbf{X}) - \mu_{t_1}(\mathbf{X}))^{2b} \right] \right)^{1/b} \quad (\text{Lemma B.1}) \\
 &\leq 2^{2(b-1)/b} \delta_{jj'}^{(a)} \left( \mathbb{E} \left[ \left( \frac{\mathbf{1}\{T=t\}}{r(t, \mathbf{X})} - 1 \right)^{2b} (f(t, \mathbf{X}) - \mu_t(\mathbf{X}))^{2b} \right] \right. \\
 &\quad \left. + \mathbb{E} \left[ \left( \frac{\mathbf{1}\{T=t_1\}}{r(t, \mathbf{X})} - 1 \right)^{2b} (f(t_1, \mathbf{X}) - \mu_{t_1}(\mathbf{X}))^{2b} \right] \right)^{1/b} \\
 &\leq 2^{2(b-1)/b} \delta_{jj'}^{(a)} \left( \mathbb{E} \left[ 2^{2b-1} \left( \frac{\mathbf{1}\{T=t\}}{r^{2b}(t, \mathbf{X})} + 1 \right) (f(t, \mathbf{X}) - \mu_t(\mathbf{X}))^{2b} \right] \right. \\
 &\quad \left. + \mathbb{E} \left[ 2^{2b-1} \left( \frac{\mathbf{1}\{T=t_1\}}{r^{2b}(t_1, \mathbf{X})} + 1 \right) (f(t_1, \mathbf{X}) - \mu_{t_1}(\mathbf{X}))^{2b} \right] \right)^{1/b} \quad (\text{Lemma B.1}) \\
 &\leq 2^{2(2b-1)/b} \delta_{jj'}^{(a)} \left( \mathbb{E} \left[ \left( \frac{1}{r^{2b-1}(t, \mathbf{X})} + 1 \right) (f(t, \mathbf{X}) - \mu_t(\mathbf{X}))^{2b} \right] \right. \\
 &\quad \left. + \mathbb{E} \left[ \left( \frac{1}{r^{2b-1}(t_1, \mathbf{X})} + 1 \right) (f(t_1, \mathbf{X}) - \mu_{t_1}(\mathbf{X}))^{2b} \right] \right)^{1/b} \\
 &\leq 2^{2(2b-1)/b} \delta_{jj'}^{(a)} \left( \frac{1}{r_{\min}^{(2b-1)/b}} + 1 \right) \left( \mathbb{E} \left[ (f(t, \mathbf{X}) - \mu_t(\mathbf{X}))^{2b} \right] \right. \\
 &\quad \left. + \mathbb{E} \left[ (f(t_1, \mathbf{X}) - \mu_{t_1}(\mathbf{X}))^{2b} \right] \right)^{1/b} \\
 &\leq 2^{2(2b-1)/b} \delta_{jj'}^{(a)} \left( \frac{1}{r_{\min}^{(2b-1)/b}} + 1 \right) \left[ \left( \mathbb{E} (f(t, \mathbf{X}) - \mu_t(\mathbf{X}))^{2b} \right)^{1/b} \right. \\
 &\quad \left. + \mathbb{E} (f(t_1, \mathbf{X}) - \mu_{t_1}(\mathbf{X}))^{2b} \right]^{1/b} \quad (\text{Subadditivity of } |x|^{1/b})
 \end{aligned} \tag{58}$$

Hence,

$$\begin{aligned}
 |\Sigma_{jj'}^{(\text{DR})}| &\leq 2^{2(2b-1)/b} \delta_{jj'}^{(a)} \left( \frac{1}{r_{\min}^{(2b-1)/b}} + 1 \right) \left[ \left( \mathbb{E} (f(t, \mathbf{X}) - \mu_t(\mathbf{X}))^{2b} \right)^{1/b} \right. \\
 &\quad \left. + \left( \mathbb{E} (f(t_1, \mathbf{X}) - \mu_{t_1}(\mathbf{X}))^{2b} \right)^{1/b} \right] + \frac{4}{r_{\min}^{(2b-1)/b}} \sigma^2 \delta_{jj'}^{(a)} \\
 &\leq 2^{2(2b-1)/b} \delta_*^{(b)} \left( \frac{1}{r_{\min}^{(2b-1)/b}} + 1 \right) \left[ \left( \mathbb{E} (f(t, \mathbf{X}) - \mu_t(\mathbf{X}))^{2b} \right)^{1/b} \right. \\
 &\quad \left. + \left( \mathbb{E} (f(t_1, \mathbf{X}) - \mu_{t_1}(\mathbf{X}))^{2b} \right)^{1/b} \right] + \frac{4}{r_{\min}^{(2b-1)/b}} \sigma^2 \delta_*^{(b)}
 \end{aligned} \tag{59}$$

We consider now  $\epsilon = b - 1 > 0$  and we assume that  $\epsilon \ll 1$ , then

$$\begin{aligned}
 &2^{2(2b-1)/b} \delta_*^{(b)} \left( \frac{1}{r_{\min}^{(2b-1)/b}} + 1 \right) \left[ \left( \mathbb{E} (f(t, \mathbf{X}) - \mu_t(\mathbf{X}))^{2b} \right)^{1/b} + \left( \mathbb{E} (f(t_1, \mathbf{X}) - \mu_{t_1}(\mathbf{X}))^{2b} \right)^{1/b} \right] + \frac{4}{r_{\min}^{(2b-1)/b}} \sigma^2 \delta_*^{(b)} \\
 &\approx 4 \delta_*^{(1+\epsilon)} \left( \frac{1}{r_{\min}^{1+\epsilon}} + 1 \right) \left( \mathbb{E} (f(t, \mathbf{X}) - \mu_t(\mathbf{X}))^2 + \mathbb{E} (f(t_1, \mathbf{X}) - \mu_{t_1}(\mathbf{X}))^2 \right) + 4 \sigma^2 \delta_*^{(1+\epsilon)} \frac{1}{r_{\min}^{1+\epsilon}}
 \end{aligned} \tag{60}$$

and, consequently,

$$|\Sigma_{jj'}^{(\text{DR})}| \leq 4 \left( \frac{1}{r_{\min}^{1+\epsilon}} C_{DR} + C_{DR}^* \right) \delta_*^{(1+\epsilon)}, \tag{61}$$

where  $C_{DR}^* = \mathbb{E} (f(t, \mathbf{X}) - \mu_t(\mathbf{X}))^2 + \mathbb{E} (f(t_1, \mathbf{X}) - \mu_{t_1}(\mathbf{X}))^2$  and  $C_{DR} = C_{DR}^* + \sigma^2$ .



### B.3 Error estimation of the X-learner.

In this case, we have  $A_t(T, \mathbf{X}) = 2 \mathbf{1}\{T = t\} - 1$  and  $B_t(T, \mathbf{X}) = (1 - \mathbf{1}\{T = t\})\mu_t(\mathbf{X}) - \mu_{t_1}(\mathbf{X}) + \sum_{t' \neq t} \mathbf{1}\{T = t'\}\mu_{t'}(\mathbf{X})$ . One can write  $\psi_t$  as

$$\begin{aligned}
 \psi_t(T, \mathbf{X}) &= A_t(T, \mathbf{X})f(T, \mathbf{X}) - \tau_t(\mathbf{x}) + B_t(T, \mathbf{X}) \\
 &= (2 \mathbf{1}\{T = t\} - 1)f(T, \mathbf{X}) - (f(t, \mathbf{X}) - f(t_1, \mathbf{X})) + (1 - \mathbf{1}\{T = t\}) \\
 &\quad \mu_t(\mathbf{X}) - \mu_{t_1}(\mathbf{X}) + \sum_{t' \neq t} \mathbf{1}\{T = t'\}\mu_{t'}(\mathbf{X}) \\
 &= (1 - \mathbf{1}\{T = t\})(\mu_t(\mathbf{X}) - f(t, \mathbf{X})) - (\mu_{t_1}(\mathbf{X}) - f(t_1, \mathbf{X})) \\
 &\quad + \sum_{t' \neq t} \mathbf{1}\{T = t'\}(\mu_{t'}(\mathbf{X}) - f(t', \mathbf{X})) = a_t + \sum_{t' \neq t} b_{t'}.
 \end{aligned} \tag{62}$$

Similarly to the M- and DR-learners calculus, and using lemma B.1:

$$\begin{aligned}
 |\mathbb{E}(f_j(\mathbf{X})f_{j'}(\mathbf{X})\psi_t^2(T, \mathbf{X}))| &\leq |\mathbb{E}(f_j^a(\mathbf{X})f_{j'}^a(\mathbf{X}))|^{1/a} \cdot |\mathbb{E}(\psi_t^{2b}(T, \mathbf{X}))|^{1/b} \\
 &\leq \delta_{jj'}^{(a)} \left| \mathbb{E}(a_t + \sum_{t' \neq t} b_{t'})^{2b} \right|^{1/b} \quad (\text{H\"older}) \\
 &\leq \delta_{jj'}^{(a)} \left( 2^{2b-1} \left( \mathbb{E}(a_t^{2b}) + \mathbb{E}(\sum_{t' \neq t} b_{t'}^{2b}) \right) \right)^{1/b} \quad (\text{Lemma B.1}) \\
 &\leq 2^{(2b-1)/b} \delta_{jj'}^{(a)} \left( \mathbb{E}(a_t^{2b}) + \mathbb{E}(\sum_{t' \neq t} b_{t'}^{2b}) \right)^{1/b} \\
 &\leq 2^{(2b-1)/b} \delta_{jj'}^{(a)} \left[ 2^{2b-1} \left( \mathbb{E}((1 - \mathbf{1}\{T = t\})^{2b} (\mu_t(\mathbf{X}) - f(t, \mathbf{X}))^{2b}) + \mathbb{E}(\mu_{t_1}(\mathbf{X}) \right. \right. \\
 &\quad \left. \left. - f(t_1, \mathbf{X}))^{2b} \right) + (K-1)^{2b-1} \sum_{t' \neq t} \mathbb{E}(\mathbf{1}\{T = t'\}(\mu_{t'}(\mathbf{X}) - f(t', \mathbf{X}))^{2b}) \right]^{1/b} \quad (\text{Lemma B.1}) \\
 &\leq 2^{(2b-1)/b} \delta_{jj'}^{(a)} \left[ 2^{2b-1} \left( \mathbb{E}(\mu_t(\mathbf{X}) - f(t, \mathbf{X}))^{2b} + \mathbb{E}(\mu_{t_1}(\mathbf{X}) - f(t_1, \mathbf{X}))^{2b} \right) \right. \\
 &\quad \left. + (K-1)^{2b-1} \sum_{t' \neq t} \mathbb{E}(\mu_{t'}(\mathbf{X}) - f(t', \mathbf{X}))^{2b} \right]^{1/b} \\
 &\leq 2^{(2b-1)/b} \delta_{jj'}^{(a)} \left[ 2^{(2b-1)/b} \left( \mathbb{E}(\mu_t(\mathbf{X}) - f(t, \mathbf{X}))^{2b} \right)^{1/b} + 2^{(2b-1)/b} \left( \mathbb{E}(\mu_{t_1}(\mathbf{X}) \right. \right. \\
 &\quad \left. \left. - f(t_1, \mathbf{X}))^{2b} \right)^{1/b} + (K-1)^{(2b-1)/b} \sum_{t' \neq t} \left( \mathbb{E}(\mu_{t'}(\mathbf{X}) - f(t', \mathbf{X}))^{2b} \right)^{1/b} \right] \\
 &\leq 2^{2(2b-1)/b} \delta_{jj'}^{(a)} \left[ \left( \mathbb{E}(\mu_t(\mathbf{X}) - f(t, \mathbf{X}))^{2b} \right)^{1/b} + \left( \mathbb{E}(\mu_{t_1}(\mathbf{X}) - f(t_1, \mathbf{X}))^{2b} \right)^{1/b} \right. \\
 &\quad \left. + \left( \frac{K-1}{2} \right)^{(2b-1)/b} \sum_{t' \neq t} \left( \mathbb{E}(\mu_{t'}(\mathbf{X}) - f(t', \mathbf{X}))^{2b} \right)^{1/b} \right]
 \end{aligned} \tag{63}$$

Given that  $\mathbb{E}(f_j(\mathbf{X})f_{j'}(\mathbf{X})A_t^2(T, \mathbf{X})) = \mathbb{E}(f_j(\mathbf{X})f_{j'}(\mathbf{X})) = \delta_{jj'}^{(1)}$ , we deduce finally

$$\begin{aligned}
 \left| \Sigma_{jj'}^{(X)} \right| &\leq \left| \mathbb{E}(f_j(\mathbf{X})f_{j'}(\mathbf{X})\psi_t^2(T, \mathbf{X})) \right| + \sigma^2 \left| \mathbb{E}(f_j(\mathbf{X})f_{j'}(\mathbf{X})A_t^2(T, \mathbf{X})) \right| \\
 &\leq 2^{2(2b-1)/b} \delta_{jj'}^{(a)} \left[ \left( \mathbb{E}(\mu_t(\mathbf{X}) - f(t, \mathbf{X}))^{2b} \right)^{1/b} + \left( \mathbb{E}(\mu_{t_1}(\mathbf{X}) - f(t_1, \mathbf{X}))^{2b} \right)^{1/b} \right. \\
 &\quad \left. + \left( \frac{K-1}{2} \right)^{(2b-1)/b} \sum_{t' \neq t} \left( \mathbb{E}(\mu_{t'}(\mathbf{X}) - f(t', \mathbf{X}))^{2b} \right)^{1/b} \right] + \sigma^2 \delta_{jj'}^{(1)} \\
 &\leq 2^{2(2b-1)/b} \delta_*^{(b)} \left[ \left( \mathbb{E}(\mu_t(\mathbf{X}) - f(t, \mathbf{X}))^{2b} \right)^{1/b} + \left( \mathbb{E}(\mu_{t_1}(\mathbf{X}) - f(t_1, \mathbf{X}))^{2b} \right)^{1/b} \right. \\
 &\quad \left. + \left( \frac{K-1}{2} \right)^{(2b-1)/b} \sum_{t' \neq t} \left( \mathbb{E}(\mu_{t'}(\mathbf{X}) - f(t', \mathbf{X}))^{2b} \right)^{1/b} \right] + \sigma^2 \delta_*^{(1)}
 \end{aligned} \tag{64}$$

where  $\delta_*^{(1)} = \max_{j,j'} \mathbb{E}(f_j(\mathbf{X})f_{j'}(\mathbf{X}))$ .

As in the previous cases, we consider now  $\epsilon = b - 1 > 0$  with  $\epsilon \ll 1$ , then

$$\begin{aligned}
 &2^{2(2b-1)/b} \delta_*^{(b)} \left[ \left( \mathbb{E}(\mu_t(\mathbf{X}) - f(t, \mathbf{X}))^{2b} \right)^{1/b} + \left( \mathbb{E}(\mu_{t_1}(\mathbf{X}) - f(t_1, \mathbf{X}))^{2b} \right)^{1/b} \right. \\
 &\quad \left. + \left( \frac{K-1}{2} \right)^{(2b-1)/b} \sum_{t' \neq t} \left( \mathbb{E}(\mu_{t'}(\mathbf{X}) - f(t', \mathbf{X}))^{2b} \right)^{1/b} \right] + \sigma^2 \delta_*^{(1)} \\
 &\approx 4 \delta_*^{(1+\epsilon)} \left( \mathbb{E}(f(t, \mathbf{X}) - \mu_t(\mathbf{X}))^2 + \mathbb{E}(f(t_1, \mathbf{X}) - \mu_{t_1}(\mathbf{X}))^2 \right) \\
 &\quad + \frac{(K-1)^2}{4} \sum_{t' \neq t} \mathbb{E}(\mu_{t'}(\mathbf{X}) - f(t', \mathbf{X}))^2 + \sigma^2 \delta_*^{(1)}.
 \end{aligned} \tag{65}$$

Therefore,

$$\left| \Sigma_{jj'}^{(X)} \right| \leq 4\delta_*^{(1+\epsilon)} C_X + \sigma^2 \delta_*^{(1)}. \tag{66}$$

where  $C_X = \mathbb{E}(f(t, \mathbf{X}) - \mu_t(\mathbf{X}))^2 + \mathbb{E}(f(t_1, \mathbf{X}) - \mu_{t_1}(\mathbf{X}))^2 + \frac{(K-1)^2}{4} \sum_{t' \neq t} \mathbb{E}(\mu_{t'}(\mathbf{X}) - f(t', \mathbf{X}))^2$ .

#### B.4 Analysis and comparison:

From equation (55), (61) and (66), one can deduce that:

- The M-learner has the largest variance and its variance upper bound is constant.
- As the term  $r_{\min}$  is present in the denominator of the upper bounds of both M-learners and DR-learners. The variance is likely to be high when there is a lack of overlap in the propensity score, i.e.  $r_{\min} \rightarrow 0$ .
- Since the upper bounds of the X-learner and DR-learner depend on the expected squared error of  $\mu_w$  (i.e.  $\mathbb{E}[f(w, \mathbf{X}) - \mu_w(\mathbf{X})]^2$ ). One can expect that, the more outcome models are precise, the lower the variance is.

**M-learner vs DR-learner.** If the potential outcome models are well-specified, then the expected squared error  $\mu_t$  is minimal and the upper bound of  $\Sigma_{jj'}^{(DR)}$  is expected to be lower for the DR-learner. One can anticipate the estimator  $\hat{\beta}_t$  of the DR-learner would have a low variance than the M-learner. Controversially, suppose the outcome models are misspecified (but the propensity score is well-specified). In that case, there is no guarantee that the DR-learner would perform better than M-learner, and it may perform even worse.

**X-learner vs M-learner.** The X-learner is likely to have low variance if the expected squared error of all outcome models  $\mu_{t'}$  is not big enough and satisfies some conditions (which is less likely to happen.)

**X-learner vs DR-learner.** It is difficult to anticipate which meta-learner would perform better in terms of variance. This will depends mainly on the expected squared error of  $\mu_{t'}$  for  $t' \in \mathcal{T}/\{t_1, t\}$ ,  $K$  and  $r_{\min}$ , whom, in some cases, will make the X-learner having less variance than the DR-learner, and the opposite in the other cases.

## C Additional details about simulated analytical functions in section 5.1.

In this section, we consider a treatment  $T$  with  $K = 10$  possible values in  $\mathcal{T} = \{t_{k+1} := \frac{k}{K-1}, k \in \{0, \dots, K-1\}\}$ , drawn from an uniform distribution, and the following outcome functions.

The linear model outcome for  $X \in \mathbb{R}$ :

$$Y(t) \mid X \sim \mathcal{N}((1+t)X, \sigma^2). \quad (67)$$

The multivariate hazard rate Imbens (2000) outcome satisfies for  $\mathbf{X} \in \mathbb{R}^5$ :

$$Y(t) \mid \mathbf{X} \sim \mathcal{N}(t + \|\mathbf{X}\| \exp(-t\|\mathbf{X}\|), \sigma^2). \quad (68)$$

We compute in the following subsections the exact components of each model: GPS  $r$ , potential outcome model  $\mu_t$  and observed outcome model  $m$ .

### C.1 The Generalized Propensity Score.

#### C.1.1 Randomized Controlled Trials (RCT) setting.

In the first design (RCT), we sample  $n$  units such that  $T$  and  $\mathbf{X}$  are independent. The true propensity score is known

$$r(t, \mathbf{X}) = \mathbb{P}(T = t) = 1/K \text{ for } t \in \mathcal{T}. \quad (69)$$

#### C.1.2 Observational non-randomized setting.

In the second design (observational studies), we combine  $K + 1$  samples in a single sample of  $n$  units. The first sample  $\mathbf{D}_K$  contains  $n_K = n/2$  units where the treatment is assigned randomly:  $\mathbf{X}$  and  $T$  are independent,  $\mathbb{P}(T = t) = 1/K$ ,  $\mathbf{X} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_5)$  when the hazard rate model is applied and  $X \sim \mathcal{U}(0, 1)$  when the linear model is applied. For  $k = 0, \dots, K - 1$  the sample  $\mathbf{D}_k$  contains  $n_k = n/(2K)$  units and  $(\mathbf{X}, T)$  is a non RCT distribution. For the linear model, the joint distribution of  $(X, T)$  is given by:

$$T = \frac{k}{K-1} \text{ and } X \text{ follows a uniform distribution } \mathcal{U}(I_k) \text{ with } I_k = \left[ \frac{k}{K}, \frac{k+1}{K} \right). \quad (70)$$

For the hazard rate model, the joint distribution of  $(\mathbf{X}, T)$  is given by:

$$T = \frac{k}{K-1} \text{ and the } X_1 \text{ follows a standardized normal law truncated Gaussian on } I_k = \left[ q_{\frac{k}{K}}, q_{\frac{k+1}{K}} \right), \quad (71)$$

where  $q_\alpha$  is the  $\alpha$ -quantile of the standardized normal law. This strategy of selecting preferentially only observations with certain characteristics is called *preferential selection* sampling and creates thus a selection bias on observed data.

For all  $k \in \{0, \dots, K - 1\}$ , the true propensity score satisfies for the linear model:

$$r(t_{k+1}, x) = \begin{cases} \frac{1+K}{2K} & \text{if } x \in I_k, \\ \frac{1}{2K} & \text{otherwise.} \end{cases} \quad (72)$$

and, for the hazard rate model, it satisfies:

$$r(t_{k+1}, \mathbf{x}) = \begin{cases} \frac{1+K}{2K} & \text{if } x_1 \in I_k, \\ \frac{1}{2K} & \text{otherwise.} \end{cases} \quad (73)$$

*Proof.* We show the proof for the hazard rate model with normal distribution. The proof remains the same for the linear model in a non-randomized setting.

Let  $A$  be a random event, then

$$\mathbb{P}(A) = \sum_{k=0}^{K-1} \frac{n_k}{n} \mathbb{P}_k(A), \quad (74)$$

where  $\mathbb{P}$  is the observed probability distribution of the combined sample and  $\mathbb{P}_k$  denotes the probability measure induced by (68), (71) and the unconfoundedness assumption 3.1.

Given the treatment  $T = t_j$  and covariate vector  $\mathbf{x} = (x, x_2, \dots, x_5)$ , we have

$$\begin{aligned}
 r(T = t_j, \mathbf{x}) &= \mathbb{P}(T = t_j \mid X_1 = x) \\
 &= \lim_{\delta \rightarrow 0} \mathbb{P}(T = t_j \mid X_1 \in [x, x + \delta]) \\
 &= \lim_{\delta \rightarrow 0} \frac{\mathbb{P}(T = t_j, X_1 \in [x, x + \delta])}{\mathbb{P}(X_1 \in [x, x + \delta])}.
 \end{aligned} \tag{75}$$

On the one hand,

$$\begin{aligned}
 \mathbb{P}(T = t_j, X_1 \in [x, x + \delta]) &= \sum_{k=0}^K \frac{n_k}{n} \mathbb{P}_k(T = t_j, X_1 \in [x, x + \delta]) \\
 &= \frac{n_j}{n} \mathbb{P}_j(T = t_j, X_1 \in [x, x + \delta]) + \frac{n_K}{n} \mathbb{P}_K(T = t_j, X_1 \in [x, x + \delta]) \\
 &= \frac{n_j}{n} \mathbb{P}_j(X_1 \in [x, x + \delta]) + \frac{n_K}{n} \mathbb{P}_K(T = 0) \mathbb{P}_K(X_1 \in [x, x + \delta]) \\
 &= \frac{1}{2K} \frac{\mathbb{P}_j(X_1 = x, X_1 \in I_j)}{\mathbb{P}_j(X \in I_j)} + \frac{1}{2K} \mathbb{P}_K(X_1 \in [x, x + \delta]) \\
 &= \frac{1}{2K} \frac{\mathbb{P}_K(X_1 \in [x, x + \delta], X_1 \in I_j)}{\mathbb{P}_K(X_1 \in I_j)} + \frac{1}{2K} \mathbb{P}_K(X_1 \in [x, x + \delta]) \\
 &= \frac{1}{2K} \frac{\mathbf{1}\{q_{j/K} - \delta \leq x < q_{(j+1)/K}\}}{\mathbb{P}_K(X_1 \in I_j)} \mathbb{P}_K(X_1 \in [x, x + \delta]) \\
 &\quad + \frac{1}{2K} \mathbb{P}_K(X_1 \in [x, x + \delta]) \\
 &= \frac{1}{2K} \left( \frac{\mathbf{1}\{q_{j/K} - \delta \leq x < q_{(j+1)/K}\}}{\mathbb{P}_K(X_1 \in I_j)} + 1 \right) \mathbb{P}_K(X_1 \in [x, x + \delta]) \\
 &= \frac{1}{2K} \left( \frac{\mathbf{1}\{q_{j/K} - \delta \leq x < q_{(j+1)/K}\}}{\Phi(q_{(j+1)/K}) - \Phi(q_{j/K})} + 1 \right) \mathbb{P}_K(X_1 \in [x, x + \delta]) \\
 &= \frac{1}{2K} \left( \frac{\mathbf{1}\{q_{j/K} - \delta \leq x < q_{(j+1)/K}\}}{(k+1)/K - k/K} + 1 \right) \mathbb{P}_K(X_1 \in [x, x + \delta]) \\
 &= \frac{1}{2K} (\mathbf{1}\{q_{j/K} - \delta \leq x < q_{(j+1)/K}\} K + 1) \mathbb{P}_K(X_1 \in [x, x + \delta]),
 \end{aligned} \tag{76}$$

where  $\Phi$  is the CDF function of the standardized normal distribution.

On the other hand,

$$\begin{aligned}
 \mathbb{P}(X_1 \in [x, x + \delta]) &= \sum_{k=0}^K \frac{n_k}{n} \mathbb{P}_k(X_1 \in [x, x + \delta]) \\
 &= \frac{1}{2K} \sum_{k=0}^{K-1} \frac{\mathbb{P}_k(X_1 \in [x, x + \delta], X_1 \in I_k)}{\mathbb{P}_k(X_1 \in I_k)} + \frac{1}{2} \mathbb{P}_K(X_1 \in [x, x + \delta]) \\
 &= \sum_{k=0}^{K-1} \frac{\mathbb{P}_K(X_1 \in [x, x + \delta])}{\mathbb{P}_K(X_1 \in I_k)} \mathbf{1}\{x \in I_k\} + \frac{1}{2} \mathbb{P}_K(X_1 \in [x, x + \delta]) \\
 &= \frac{1}{2K} \left( \sum_{k=0}^{K-1} \frac{\mathbf{1}\{q_{k/K} - \delta \leq x < q_{(k+1)/K}\}}{\mathbb{P}_K(X_1 \in I_k)} + K \right) \mathbb{P}_K(X_1 \in [x, x + \delta]) \\
 &= \frac{1}{2K} \left( \sum_{k=0}^{K-1} K \mathbf{1}\{q_{k/K} - \delta \leq x < q_{(k+1)/K}\} + K \right) \mathbb{P}_K(X_1 \in [x, x + \delta]).
 \end{aligned} \tag{77}$$

Finally,

$$\begin{aligned}
 r(t_j, \mathbf{x}) &= \lim_{\delta \rightarrow 0} \frac{\mathbb{P}(T = t_j, X_1 \in [x, x + \delta])}{\mathbb{P}(X_1 \in [x, x + \delta])} \\
 &= \lim_{\delta \rightarrow 0} \frac{\frac{1}{2K} (\mathbf{1}\{q_j/K - \delta \leq x < q_{(j+1)/K}\}K + 1) \mathbb{P}_K(X_1 \in [x, x + \delta])}{\frac{1}{2K} \left( \sum_{k=0}^{K-1} \mathbf{1}\{q_k/K - \delta \leq x < q_{(k+1)/K}\}K + K \right) \mathbb{P}_K(X_1 \in [x, x + \delta])} \\
 &= \lim_{\delta \rightarrow 0} \frac{\mathbf{1}\{q_j/K - \delta \leq x < q_{(j+1)/K}\}K + 1}{\sum_{k=0}^{K-1} \mathbf{1}\{q_k/K - \delta \leq x < q_{(k+1)/K}\}K + K} \\
 &= \frac{\mathbf{1}\{x \in I_j\}K + 1}{\left( \sum_{k=0}^{K-1} \mathbf{1}\{x \in I_k\} + 1 \right) K} \\
 &= \begin{cases} \frac{1+K}{2K} & \text{if } x \in I_j, \\ \frac{1}{2K} & \text{otherwise.} \end{cases}
 \end{aligned} \tag{78}$$

□

## C.2 The potential outcome models.

The potential outcome models are given directly by the conditional mean. For the linear model,  $\mu_t$  satisfies for all  $t \in \mathcal{T}$ :

$$\mu_t(x) = (1 + t)x, \tag{79}$$

and, for the hazard rate model,  $\mu_t$  is given by:

$$\mu_t(\mathbf{x}) = t + \|\mathbf{x}\| \exp(-t\|\mathbf{x}\|). \tag{80}$$

## C.3 The observed outcome models.

For the linear model, the observed outcome model  $m$  can be computed as:

$$\begin{aligned}
 m(x) &= \mathbb{E}(Y_{\text{obs}} \mid X = x) \\
 &= \mathbb{E}((1 + T)\mathbf{X} \mid X = x) \\
 &= (1 + \mathbb{E}(T \mid X = x))x \\
 &= \left(1 + \sum_{k=1}^K r(t_k, x)t_k\right)x,
 \end{aligned} \tag{81}$$

where  $r$  is given by (72).

and, for the hazard rate model,  $m$  can be computed as:

$$\begin{aligned}
 m(\mathbf{x}) &= \mathbb{E}(\mathbb{E}(Y_{\text{obs}} \mid \mathbf{X}, T) \mid \mathbf{X} = \mathbf{x}) \\
 &= \mathbb{E}(T + \|\mathbf{X}\| \exp(-T\|\mathbf{X}\|) \mid \mathbf{X} = \mathbf{x}) \\
 &= \mathbb{E}(T \mid \mathbf{X} = \mathbf{x}) + \|\mathbf{x}\| \mathbb{E}(\exp(-T\|\mathbf{X}\|) \mid \mathbf{X} = \mathbf{x}) \\
 &= \sum_{k=1}^K r(t_k, \mathbf{x})t_k + \sum_{k=1}^K \|\mathbf{x}\| r(t_k, \mathbf{x}) \exp(-t_k\|\mathbf{x}\|).
 \end{aligned} \tag{82}$$

where  $r$  is given by (73).

## D Additional numerical results and plots.

### D.1 linear model in randomized setting.

 Table 4: **mPEHE** for three different ML base-learners; Case where nuisance components are exact.

| Meta-learner | XGBoost      | RandomForest         | Linear Model                           |
|--------------|--------------|----------------------|--|
| M-Learner    | 2.248        | 2.07                 | 0.099                                  |
| DR-Learner   | 0.159        | 0.134                | $7.04 \cdot 10^{-3}$                   |
| X-Learner    | <b>0.022</b> | <b>0.028</b>         | <b><math>1.53 \cdot 10^{-3}</math></b> |
| RLin-Learner |              | $7.33 \cdot 10^{-3}$ |  |

 Table 5: **mPEHE** for three different ML base-learners; Case when nuisance components are well-specified.

| Meta-learner | XGBoost       | RandomForest  | Linear Model                         |
|--------------|---------------|---------------|--------------------------------------|
| T-Learner    | 0.061         | <b>0.037</b>  | $7.37 \cdot 10^{-3}$                 |
| S-Learner    | <b>0.029</b>  | 0.040         | $3.65 \cdot 10^{-3}$                 |
| M-Learner    | 1.23          | 1.15          | 0.210                                |
| DR-Learner   | 0.063 (0.063) | (0.060) 0.060 | $7.22 \cdot 10^{-3}$ ( <b>3.39</b> ) |
| X-Learner    | 0.059 (0.030) | (0.041) 0.079 | $7.36 \cdot 10^{-3}$ (3.59)          |
| RLin-Learner | 0.122         | 0.112         | 0.046                                |

 Table 6: **mPEHE** for three different ML base-learners; Case when the propensity score is misspecified.

| Meta-learner | XGBoost      | RandomForest | Linear Model         |
|--------------|--------------|--------------|----------------------|
| M-Learner    | 3.54         | 3.31         | 1.31                 |
| DR-Learner   | <b>0.119</b> | <b>0.104</b> | <b>0.011</b>         |
| X-Learner    | 0.030        | 0.041        | $3.59 \cdot 10^{-3}$ |
| RLin-Learner | 0.318        | 0.313        | 0.334                |

 Table 7: **mPEHE** for three different ML base-learners; Case when the outcome models are misspecified.

| Meta-learner | XGBoost | RandomForest | Linear Model |
|--------------|---------|--------------|--------------|
| M-Learner    | 1.23    | 1.15         | 0.210        |
| DR-Learner   | 0.737   | 0.800        | 0.217        |
| X-Learner    | 0.282   | 0.282        | 0.246        |
| RLin-Learner |         | <b>0.045</b> |              |

 Table 8: **mPEHE** for three different ML base-learners; Case when nuisance components are misspecified.

| Meta-learner | XGBoost | RandomForest | Linear Model |
|--------------|---------|--------------|--------------|
| M-Learner    | 3.54    | 3.31         | 1.31         |
| DR-Learner   | 1.66    | 1.85         | 0.758        |
| X-Learner    | 0.282   | 0.282        | <b>0.246</b> |
| RLin-Learner |         | <b>0.280</b> |              |

## D.2 linear model in non-randomized setting

 Table 9: **mPEHE** for three different ML base-learners; Case where nuisance components are exact.

| Meta-learner | XGBoost              | RandomForest | Linear Model                           |
|--------------|----------------------|--------------|--|
| M-Learner    | 3.68                 | 2.33         | 0.68                                   |
| DR-Learner   | 0.287                | 0.147        | 0.014                                  |
| X-Learner    | <b>0.023</b>         | <b>0.030</b> | <b><math>1.57 \cdot 10^{-3}</math></b> |
| RLin-Learner | $9.44 \cdot 10^{-3}$ |              |  |

 Table 10: **mPEHE** for three different ML base-learners; Case when nuisance components are well-specified.

| Meta-learner | XGBoost       | RandomForest  | Linear Model                           |
|--------------|---------------|---------------|--|
| T-Learner    | 0.061         | <b>0.042</b>  | $7.37 \cdot 10^{-3}$                   |
| RegT-Learner | 0.052         | <b>0.042</b>  | $7.60 \cdot 10^{-3}$                   |
| S-Learner    | <b>0.029</b>  | 0.050         | <b><math>3.65 \cdot 10^{-3}</math></b> |
| M-Learner    | 1.23          | 1.15          | 0.209                                  |
| DR-Learner   | 0.060 (0.055) | (0.068) 0.095 | $7.60 (3.95) \cdot 10^{-3}$            |
| X-Learner    | 0.051 (0.030) | (0.045) 0.079 | $7.33 (3.95) \cdot 10^{-3}$            |
| RLin-Learner | 0.122         | 0.127         | 0.046                                  |

## D.3 Hazard rate model in randomized setting

 Table 11: **mPEHE** for three different ML base-learners; Case where nuisance components are exact.

| Meta-learner | XGBoost      | RandomForest | Linear Model |
|--------------|--------------|--------------|--------------|
| M-Learner    | 4.25         | 4.22         | 0.52         |
| DR-Learner   | 0.127        | 0.139        | 0.099        |
| X-Learner    | <b>0.045</b> | <b>0.085</b> | <b>0.098</b> |
| RLin-Learner | 0.100        |              |              |

 Table 12: **mPEHE** for three different ML base-learners; Case when nuisance components are well-specified.

| Meta-learner | XGBoost              | RandomForest  | Linear Model         |
|--------------|----------------------|---------------|----------------------|
| T-Learner    | 0.171                | 0.267         | <b>0.105</b>         |
| S-Learner    | 0.154                | 0.267         | 0.649                |
| M-Learner    | 1.52                 | 1.76          | 0.792                |
| DR-Learner   | (0.154) 0.163        | 0.286 (0.282) | (0.106) 0.461        |
| X-Learner    | <b>(0.149)</b> 0.161 | (0.284) 0.285 | <b>(0.105)</b> 0.637 |
| RLin-Learner | 0.227                | <b>0.241</b>  | 0.691                |

#### D.4 Hazard rate model in non-randomized setting

 Table 13: **mPEHE** for three different ML base-learners; Case where nuisance components are exact.

| Meta-learner | XGBoost      | RandomForest | Linear Model |
|--------------|--------------|--------------|--------------|
| M-Learner    | 6.33         | 5.81         | 3.52         |
| DR-Learner   | 0.138        | 0.140        | 0.100        |
| X-Learner    | <b>0.044</b> | <b>0.085</b> | <b>0.098</b> |
| RLin-Learner |              | 0.290        |              |

 Table 14: **mPEHE** for three different ML base-learners; Case when nuisance components are well-specified.

| Meta-learner         | XGboost              | RandomForest  | Linear Model         |
|----------------------|----------------------|---------------|----------------------|
| T-Learner            | 0.184                | 0.251         | 0.128                |
| <i>Reg</i> T-Learner | 0.158                | 0.253         | <b>0.111</b>         |
| S-Learner            | 0.166                | 0.269         | 0.642                |
| M-Learner            | 1.56                 | 1.55          | 0.866                |
| DR-Learner           | (0.151) 0.171        | (0.275) 0.288 | <b>(0.111)</b> 0.495 |
| X-Learner            | <b>(0.149)</b> 0.162 | (0.270) 0.286 | (0.114) 0.627        |
| RLin-Learner         | 0.235                | <b>0.178</b>  | 1.00                 |



D.5 Asymptotic performances when  $n$  and  $K$  increase.

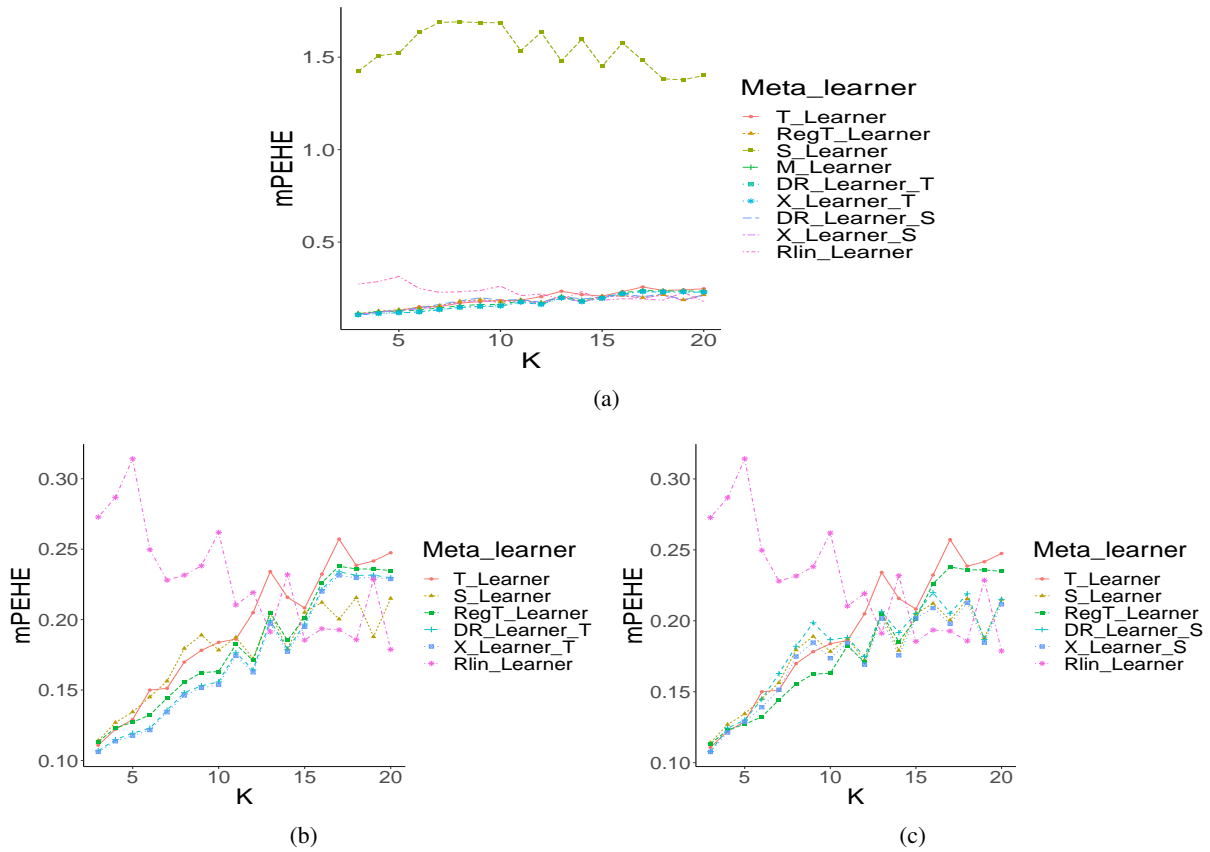


Figure 2: Variation of meta-learner's performances when number of possible treatment values  $K$  for the hazard rate function in observational design setting. (a): All meta-learners; (b): When the potential outcome models  $\mu_s$  are estimated by regT-learning; (c): When the potential outcome models  $\mu_s$  are estimated by S-learning.

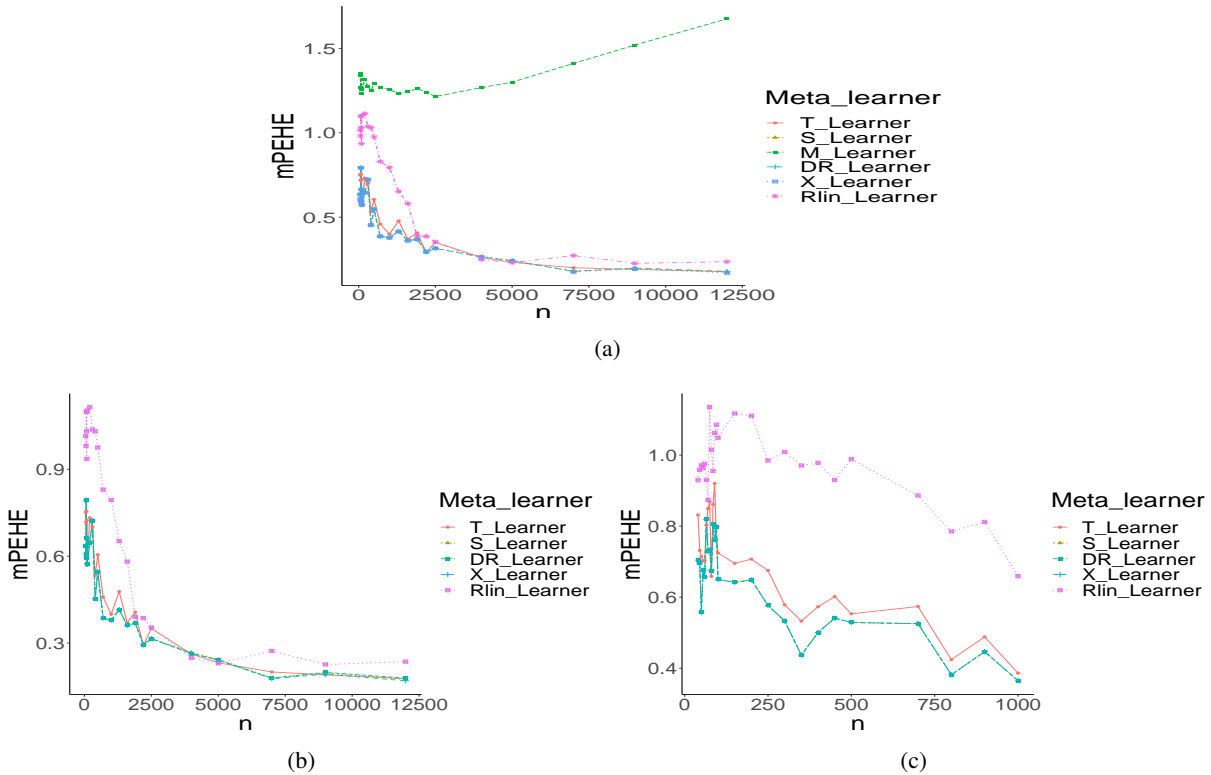


Figure 3: Variation of meta-learner's performances with the observed sample size  $n$  for the hazard rate function in observational design setting. (a): All meta-learners; (b): Without the M-learner; (c): Without the M-learner with a focus on low sample regime.

## E Description of the semi-synthetic dataset

This section describes the process of generating our semi-synthetic dataset simulating the heat delivered by a multistage fracturing EGS (see Figure 4). The process involved the creation of a conceptual reservoir model and modelling multiple well's completion scenarios. The output (heat extraction performance) obtained from physics-based simulation experiments was tabulated with inputs in the semi-synthetic dataset.

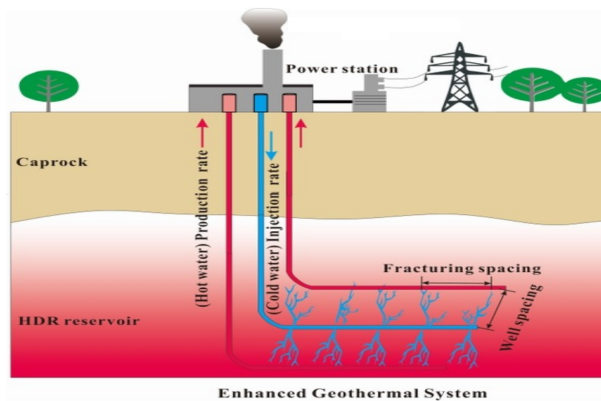


Figure 4: An illustration of an multistage fracturing EGS. The objective is to evaluate the impact of lateral length of the well on the heat extraction performance.

To ease confidentiality and non-disclosure information issues, input data for the model was fabricated. However, data has been selected from reliable sources such as, field observations, journals and books to be within the range of interest. This allowed building a plain but representative reservoir model that would provide realistic results of an EGS.

The heat extraction performance from a single fracture ( $Q_{fracture}$ ) is determined using fracture length, fracture height, fracture width, fracture permeability, reservoir porosity, reservoir permeability and pore pressure. Modelling and simulation work were done using preprocessor and reservoir simulation tools (*Petrel/Eclipse*).

The four physical properties of the fracture were investigated, and the list of values used for each parameter can be observed in Table 15. In the end,  $10 \times 10 \times 2 \times 3 = 600$  fracture's simulation cases has been realized.

Table 15: Fracture parameters and their range of variation for simulations

| Variable                   | Range of variation               |
|----------------------------|----------------------------------|
| Fracture length (ft)       | [100, 1000] by a step of 100 ft. |
| Fracture height (ft)       | [50, 500] by a step of 50 ft.    |
| Fracture width (in)        | {0.1, 0.2}                       |
| Fracture Permeability (md) | {30000, 85000, 19000}            |

To emulate distinct reservoir schemes, it was decided to vary three main parameters; porosity, permeability and pore pressure. For porosity and permeability, the simulator tool took the minimum and maximum values and estimated the physical properties across the reservoir. Three different multipliers were applied to define three (Low, Base and High) scenarios. Concerning pore pressure, three specific values were defined to simulate under-normal, normal (base) and overpressure (high) gradient conditions. Therefore,  $3 \times 3 \times 3 = 27$  possible scenarios were defined. Table 16 displays the range of minimum and maximum values for the three reservoir properties to be varied.

Table 16: Reservoir parameters and their range of variation for simulations.

| Variable                       | Range of variation                                 |
|--------------------------------|--|
| $(K_{min}, K_{max})$ (md)      | {(0.0054, 0.0157), (0.054, 0.157), (0.109, 0.314)} |
| $(Por_{min}, Por_{max})$ (dec) | {(0.0054, 0.0157), (0.054, 0.157), (0.109, 0.314)} |
| Pore pressure (psi)            | {5000, 7000, 9000}                                 |

By combining different reservoir scenarios with single fracture simulations, we obtained a single dataset with 16,200 possible cases for a fracture in a reservoir then we simulated the heat extraction performance for each experiment. Simulation's results were tabulated in the dataset "*Single\_Fracture\_Simulation\_Cases\_16200.csv*".

The next step is to define well characteristics (lateral lengths and fracture spacing) to evaluate the heat extraction performance of the well, when reservoir and fracture properties were not changed.

Table 17: Well parameters and their range of variation.

| Variable              | Range of variation                  |
|-----------------------|-------------------------------------|
| Lateral length (ft)   | [2000, 14000] by a step of 1000 ft. |
| Fracture spacing (ft) | [100, 500] by a step of 100 ft.     |

Regarding the spacing efficiency coefficient, this coefficient was used to model interactions between fractures and penalize the heat extraction performance of a single fracture in the presence of other close fractures, that is, when the spacing between two fractures is small. Indeed, if the fractures are spaced too close to each other, there may not be enough thermal energy in the rock to heat the water, which decreases the heat extraction efficiency. Modelling this efficiency led to the efficiency table "*Fracture\_Efficiency.csv*" that describe what would be the well's heat performance behaviour with respect to the fracture spacing selected. Based on this table, one can interpolate the efficiency to draw the curve (see Figure 5) and thus obtain the spacing efficiency coefficient for any desired value fracture spacing.

The final generation of the semi-synthetic dataset "*Main\_Dataset.csv*" was achieved by combining two main tables created using R programming language. This table allowed to calculation the heat performance of a well for any lateral

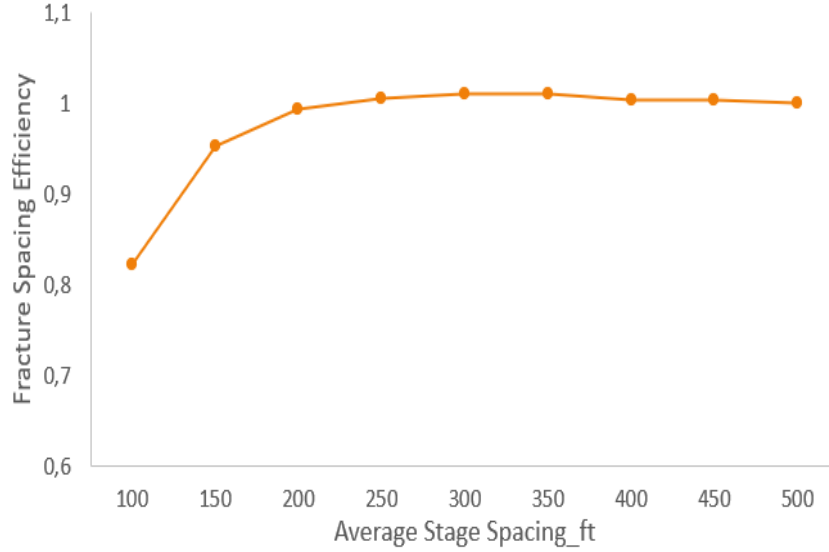


Figure 5: Cross plot between fracture spacing efficiency and average stage spacing.

length and fracture spacing between 500 ft and 100 ft with the associated spacing efficiency coefficient defined in the efficiency table, following the physical model  $Q_{well}(\ell_L) = Q_{fracture} \times \ell_L/d \times \eta_d$ .

The three datasets are available in the zip file in Supplementary Materials "*Semi-synthetic-EGS.zip*". They will also be shared in the following repository for public use.

Finally, we emphasize that the designed methodology applied for this study focused not only on generating a semi-synthetic dataset using reservoir numerical simulation but also on creating a new benchmarking dataset for the comparison and validation of causal inference methods. Indeed, following the last step of creating the final dataset "*Main\_Dataset.csv*", any user can define different distributions (with different values) on lateral lengths in the range [2000, 14000] and fracture spacing in range [100, 500], pick-up the corresponding spacing efficiency coefficients using the curve drawn in Figure 5 and generate a new semi-synthetic dataset by extrapolating them with "*Single\_Fracture\_Simulation\_Cases\_16200.csv*" dataset.

**Creating a non-randomized biased dataset.** The idea of this step was to create a collection of biased data from the main semi-synthetic dataset to emulate observational data found in real-world situations. For example, geothermal wells with larger lateral lengths are likely to have more fractures (expensive wells are located in better geological areas). The opposite is seen for smaller wells that tend to be associated with less fractures. This situation creates a discrepancy between what engineers expect with physical models and what they observe in the field data. The biased data, with 9,992 observations, was generated by following the *preferential selection* strategy from the main dataset. Figure 6 shows the difference between the *real* heat extraction performance of the EGS and the observed heat extraction performance on the field: low (under-estimated) heat performance for small wells and high (over-estimated) heat performance for large wells

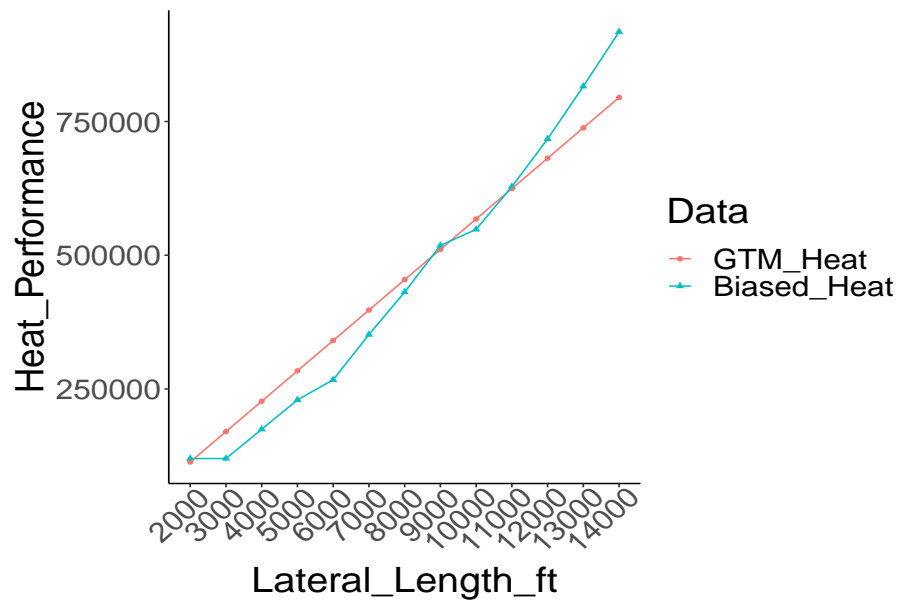


Figure 6: An illustration of selection bias on the heat performance. Red line: The heat extraction performance on the main dataset (i.e. Ground Truth Model). Blue line: The heat performance on the biased dataset (i.e. observed response).